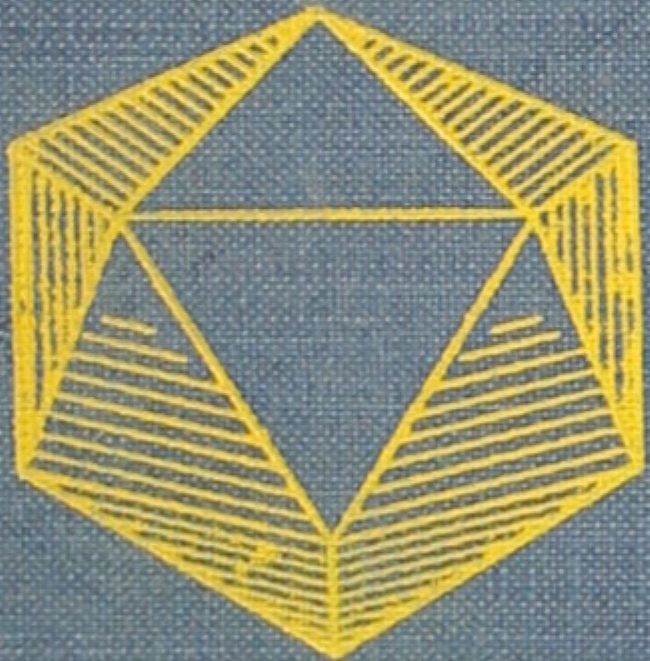
DYNAMIC SOLID GEOMETRY



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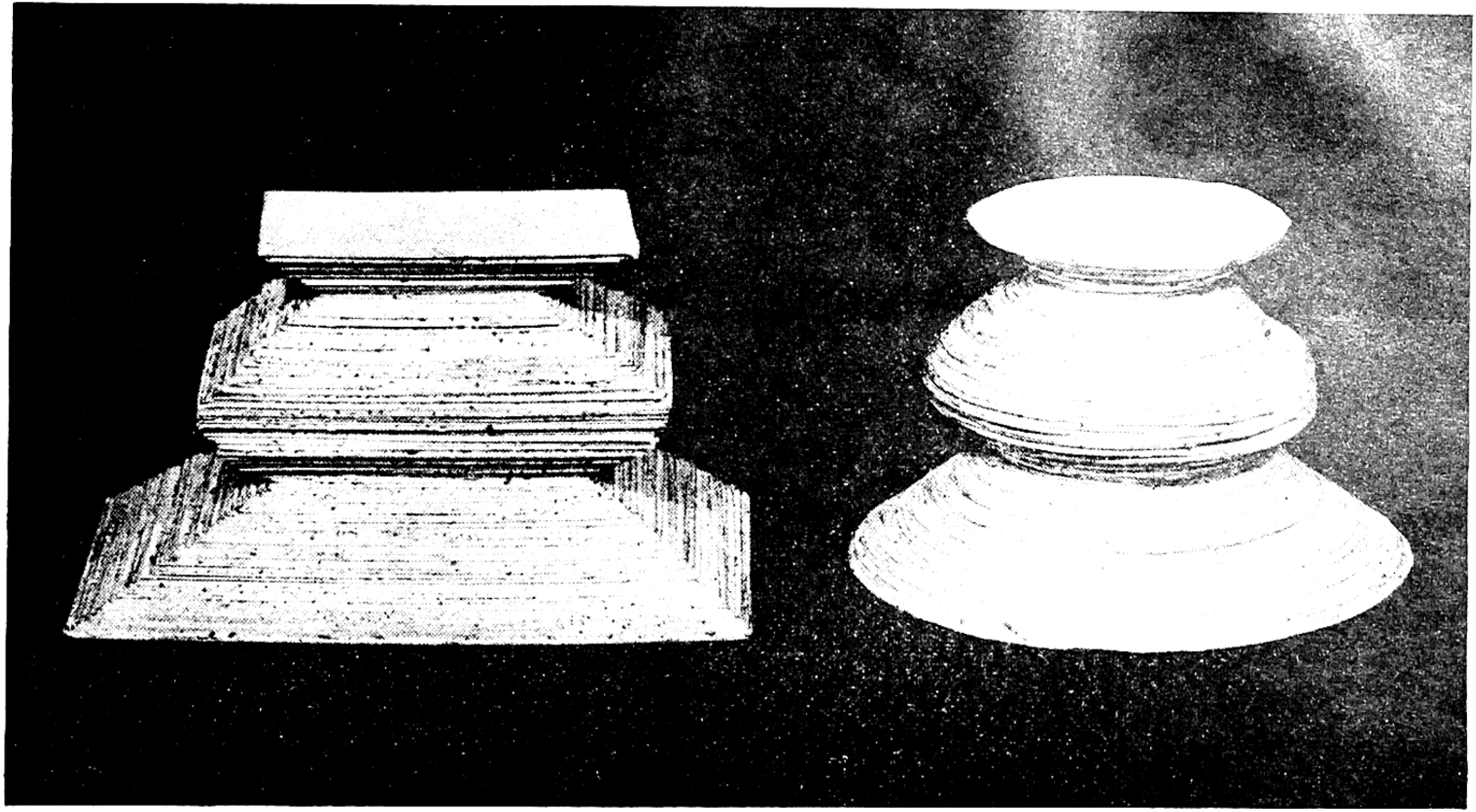

A student's model of Cavalieri's solids. Each of the solids is made up of two hundred cards, the area of each rectangular card being equal to that of the corresponding circular card. See page 98.

DYNAMIC SOLID GEOMETRY

S/c

By

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PRINCETON, NEW JERSEY

TORONTO

LONDON

NEW YORK

D. VAN NOSTRAND COMPANY, INC.
120 Alexander St., Princeton, New Jersey
257 Fourth Avenue, New York 10, New York
25 Hollinger Rd., Toronto 16, Canada

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Library of Congress Catalogue Card No. 52–6384

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Preface

In DYNAMIC SOLID GEOMETRY the informal approach through motion and stress on large over-all concepts used in *Dynamic Plane Geometry* are retained. However, students who have studied any good text in plane geometry are prepared to use this book. Following are some of its features.

Organization. The material is organized into large units instead of being presented a proposition at a time. From the student's point of view, the difficulty with solid geometry has been that theorems keep coming from apparently nowhere. The student either will not or can not organize these propositions into a whole. In DYNAMIC SOLID GEOMETRY the five chapters are organized into thirty-six lessons, five of which are reviews. Each lesson is a complete phase of a chapter. At or near the beginning of the lesson, the student is given a bird's-eye view of the unit and is led to expect certain necessary propositions and to appreciate their purpose. As explained below, this does not mean that the student will have to learn a unit that is too large for him; most of the lessons will require more than one day to complete.

Motion. The concept of change and motion is often used to relate theorems to one another as well as to bring about a better understanding of individual propositions. This, again, enables the student to see a proposition as one phase of a general pattern.

Summaries. The chapter summaries and reviews are further designed to make the student see the unit he has studied from a broad point of view. Thus, instead of remembering individual theorems, he is helped to assimilate generalized concepts.

Flexibility. Solid geometry is currently being taught in three forms, which may be characterized as follows: (1) The logical structure and the applications receive about equal treatment; (2) proofs of propositions are not required, but much time is spent on originals and numerical problems; and (3) a course of less than one semester which places the whole stress on numerical exercises, the necessary theory being developed intuitively. DYNAMIC SOLID GEOMETRY is designed for all three courses. In all courses it is recommended that the text of each lesson be first read without detailed proofs. Along with the assignment of the exercises the detailed proofs involved in Course 1 may be included. (The exercises are so arranged that, if desired, some of them may be assigned before completing the text of the lesson.) In the abbreviated course, the theory can be developed by using the informal part of the text as introductory and connective material. This material is really a "running story" of solid geometry. It gives the student an understanding of relationships in general space and enables him to solve problems — without delving into all the formal proofs.

To save time for the teacher, the text and exercises most valuable for this course are suggested on pages 217–218. Teachers will appreciate the large number of varied and graded numerical exercises.

Transition from plane geometry. In many cases, students begin their study of solid geometry after they have been away from plane geometry for a year and a half. In this book the underlying concepts and methods of plane geometry are carefully reviewed. Important propositions are illustrated by figures in perspective, thus serving the double purpose of reviewing plane geometry and of introducing figures in general space.

Stress on reasoning. The attention given to methods of drawing conclusions in plane geometry is continued and refined, especially on the superior level.

Space analogy. The constant comparison between plane and solid geometry is used to develop a better understanding of each. The more complete treatment of spherical geometry illustrates this feature.

Exploration and discovery. The student is taught to use experimentation, analogy, and dynamic exploration (mostly in the form of exercises) to discover relationships in space. At the same time he learns that in general space intuition is to be trusted even less than in plane geometry; that his conclusions should, whenever possible, be made to rest on deductive reasoning.

Correlation with algebra. Following the pattern in *Dynamic Plane Geometry*, solid geometry has been tied more closely with algebra. For example, on the superior level mathematical induction is used to prove geometric propositions. Some conventional demonstrations have been given algebraic form.

Fresh approach. Many proofs have been simplified. Fresh applications that appeal to the type of student studying solid geometry have been introduced. New concepts are not merely stated; they are accompanied by some suggested activity for the student. All these are designed to give the student a broader understanding of the world through the study of solid geometry.

DAVID SKOLNIK

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SYMBOLS AND ABBREVIATIONS

SYMBOLS

∠, ∠s	angle, angles
⌒	arc
⊙, ⊙s	circle, circles
≅	congruent, is congruent to
°	degree
=	equals, is (are) equal to
≠	unequal, is (are) not equal to
>	is (are) greater than
<	is (are) less than
≗	is measured by
′	minute *or* prime *or* foot
″	second *or* inch
∥	parallel, is parallel to
∥s	parallels
∦	not parallel
▱, ▱s	parallelogram, parallelograms
▭, ▭s	rectangle, rectangles
⊥, ⊥s	perpendicular(s), is perpendicular to
∼	similar, is (are) similar to
△, △s	triangle, triangles

ABBREVIATIONS

adj.	adjacent
alt.	altitude
alt. int.	alternate interior
ax.	axiom
comp.	complementary
cons.	construct *or* construction
corr.	corresponding
def.	definition
diag.	diagonal
diam.	diameter
dihed.	dihedral
ext.	exterior
int.	interior
isos.	isosceles
opp.	opposite
polyhed.	polyhedral
post.	postulate
prop.	proposition
pt.	point
quad.	quadrilateral
rect.	rectangle
req.	required
rt.	right
sq.	square
st.	straight
subs.	substitution
subt.	subtraction
sup.	supplementary
tan.	tangent
trans.	transversal
trap.	trapezoid
trihed.	trihedral
vert.	vertical

1 Emerging from the Plane

Figures in Space

PLANE GEOMETRY

1. Geometry is a study of space. In plane geometry we were limited to a space like the surface of the blackboard which we call a plane. In solid geometry we shall study figures that are not so confined. You are already familiar with the shapes of many of these figures, such as the cube, the pyramid, the sphere, the cylinder, and the cone. We may think of the plane as a cross section of the more general space that we are about to study; or we may think of the study of general space as an extension of the study of the plane.

Fundamental Concepts in Plane Geometry

2. Since the study of solid geometry is based on the study of plane geometry, we begin by reviewing some basic concepts of the latter. On page 207 you will find a summary of important facts in plane geometry. In this lesson we are reviewing the fundamental and unique ideas involved in arriving at these facts.

Underlying all our thinking in geometry, as well as thinking in general, are two methods of drawing conclusions. They are the inductive method and the deductive method. In the *inductive* method, we draw a general conclusion by observing a number of particular facts. Usually, these facts are found by observation or by more elaborate measurements. Thus, our experience with space teaches us that through any two particular points one line, and only one line, can be drawn. We generalize this by saying that two points determine a straight line. We sometimes call this an *intuitive* fact, or a fact arrived at by intuition. In the *deductive* method of reasoning, we arrive at a proposition as a necessary conclusion from a more general statement.

Although the inductive method is important in plane geometry, and will be even more important in solid geometry, the mathematical facts in both kinds of geometry are made secure and finally settled by deductive proof. We shall therefore mention the essentials of this kind of proof. They are certain terms and statements that we accept without definition or proof. Such terms are called undefined terms, and the statements are called assumed propositions or postulates. Using undefined terms, we define other terms; and using unproved propositions, we prove other propositions that we call *theorems* or *corollaries*.

The exercises that follow will help you recall other fundamental ideas about plane geometry. Still others will be reviewed when we are ready to apply them again in solid geometry.

EXERCISES

When given facts always lead to certain other facts as the only possible conclusion, we say that the given facts determine the conclusion. Which of the following statements about determining a conclusion are true and which are false?

1. **a.** Two nonparallel lines determine a point of intersection.
  b. Through a given point, a perpendicular to a given line is determined.
  c. Through a given point, a parallel to a given line is determined.
  d. Three points determine a circle through them.

2. **a.** Two points, each equidistant from the ends of a line segment, determine the perpendicular bisector of the line segment.
  b. Two nonintersecting lines determine a common perpendicular to these lines.
  c. Two intersecting lines determine a common perpendicular to these lines.
  d. A quadrilateral determines a circle circumscribed about it.

3. **a.** The equation $x + y = 10$ determines the numerical value of x.
  b. The fraction $\frac{x}{y}$ is determined for all values of x and y.
  c. The equation $7 + \sqrt{x} = 5$ determines the value of x.
  d. The equation $x(x - 2) + 1 = (x - 1)^2$ determines the value of x.

When a conclusion is determined by given facts, called the hypothesis, we say that the conclusion is necessary to the hypothesis. In the following statements, write the conclusion for each hypothesis. (You may refer to the summary of plane geometry on page 207.)

4. **a.** If two lines do not intersect, they are —?—.
  b. If two lines are perpendicular to the same line, they are —?—.
  c. If two lines have no common perpendicular, they —?—.
  d. If a line is perpendicular to one of two intersecting lines, it is —?—.

5. **a.** If a figure contains all the points at a given distance from a given point and no others, it is a(n) —?—.
  b. If a figure contains all the points equidistant from the sides of an angle, it is a(n) —?—.
  c. If three rays issue from the same point, then —?— angles are formed.
  d. If two adjacent angles are supplementary, their exterior sides —?—.

6. a. If the opposite sides of a quadrilateral are equal, the figure is a(n) —?—.
   b. In a quadrilateral, the sum of the angles equals —?—.
   c. If the opposite sides of a quadrilateral are equal, the opposite angles are —?—.
   d. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a(n) —?—.

The converse of a proposition is formed by interchanging a condition in the hypothesis with a condition in the conclusion. If there is only one condition in the hypothesis and only one in the conclusion, only one converse is possible. Write the converse of each of the following propositions. State whether the original proposition or the converse or both are valid (true).

7. a. If two angles are complementary, they are both acute angles.
   b. Two right angles are supplementary.
   c. One-half an obtuse angle is an acute angle.
   d. If one of two supplementary angles is obtuse, the other is acute.

8. a. If a triangle is scalene, it is not equilateral.
   b. If the square of one side of a triangle equals the sum of the squares of the other two sides, the triangle is a right triangle.
   c. If a figure is not a square, it is not a rectangle.
   d. If a figure is not a rhombus, it is not a square.

9. a. If two parallel lines are cut by a transversal, the alternate interior angles are equal.
   b. If one transversal of two lines makes the corresponding angles equal, any other transversal will do likewise.
   c. If two lines are cut by a transversal and the interior angles on the same side of the transversal are equal, the lines are not parallel.
   d. If a line is perpendicular to one of two parallels, it is perpendicular to the other also.

10. Form the converses of the propositions in Exercise 4 and tell if they are valid.

11. Form the converses of the propositions in Exercise 5 and tell if they are valid.

12. Form the converses of the propositions in Exercise 6 and tell if they are valid.

13. In plane geometry you learned that a definition must be reversible. This means that a definition contains two "if-then" propositions. Thus the definition of supplementary angles means:
    1. If the sum of two angles is a straight angle, the angles are supplementary and
    2. If two angles are supplementary, their sum is a straight angle.

State the two "if-then" propositions implied in each of the following definitions. (If necessary refer to the plane geometry section on pages 207–216.)

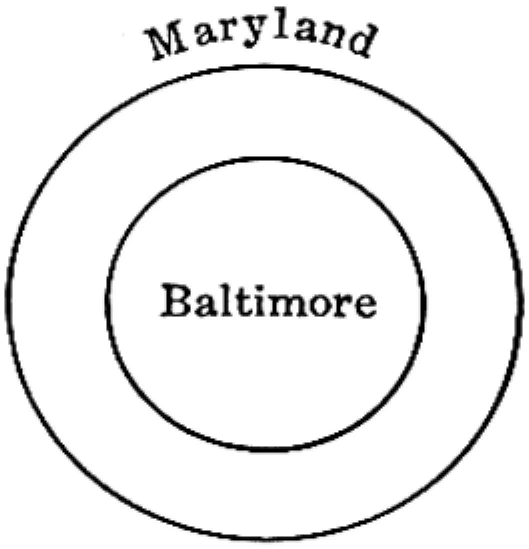
a. An acute triangle.
b. A parallelogram.
c. A rectangle.
d. A diameter of a circle.
e. A tangent to a circle.
f. An inscribed angle.

There are two kinds of propositions: those that are assumed and those that are proved. The assumed propositions are in the form of postulates, axioms, or definitions. The propositions that are proved are called *theorems;* if their proofs are simple, they are called corollaries. From each of the propositions stated below, a new theorem or corollary may be derived, as suggested. Complete this new proposition.

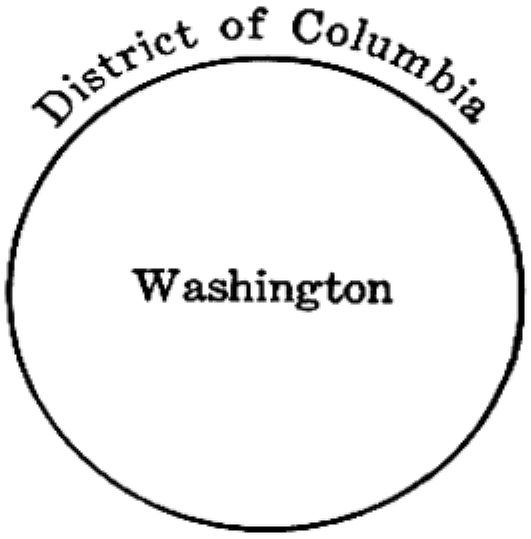
14. a. *Postulate.* Two points determine a straight line: Two lines can intersect in —?—.
b. *Postulate.* A straight line is the shortest distance between two points: The sum of two sides of a triangle is —?—.
c. Definition of a rectangle (§ 395): All the angles of a rectangle —?—.
d. Definition of a rhombus: All the sides of a rhombus —?—.

SUPERIOR WORK

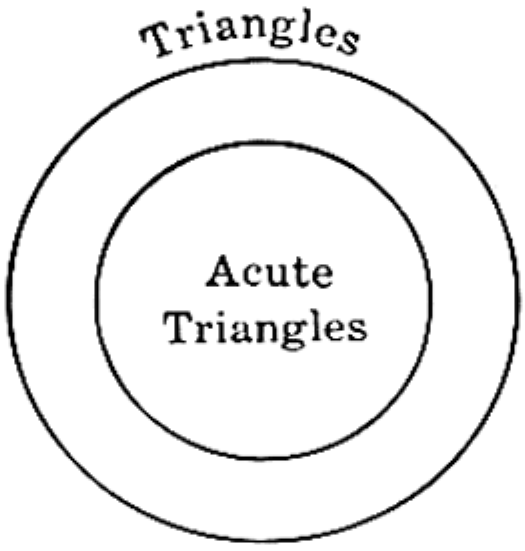
S 1. An "if-then" proposition may be illustrated by a circle within a circle. The figure illustrates the conditional proposition that, if a person lives in Baltimore, then he lives in Maryland. The figure also shows that there are people living in Maryland who do not live in Baltimore, which means that the converse of the proposition is not true.



Now consider this proposition: If a person lives in Washington, he lives in the District of Columbia. Since Washington occupies all of the District of Columbia, the circles illustrating this proposition coincide. There are no inhabitants in the District of Columbia outside Washington, so the converse is true. The proposition is therefore reversible. This is an important require-



ment of a definition. Because our proposition has that requirement, it may be used to define a resident of Washington without further qualification.



Usually a term is defined by first stating that it belongs to the next higher group. Thus, an acute triangle is defined in terms of triangles in general. The boundary of the inner circle illustrating this definition is then limited by the fact that the triangles in that circle have three acute angles, whereas the rest have only two.

Draw circles to illustrate the other definitions in Exercise 13.

S 2. If a proposition has more than one condition in the hypothesis or in the conclusion, then we may have more than one converse. We may form a new converse by interchanging one condition in the hypothesis with one condition in the conclusion.

Find propositions in Exercises 4, 6, and 7 in which the hypothesis may be broken up into two conditions. Form the converses by interchanging one of these conditions at a time with the conclusion and tell which ones are valid.

PERSPECTIVE

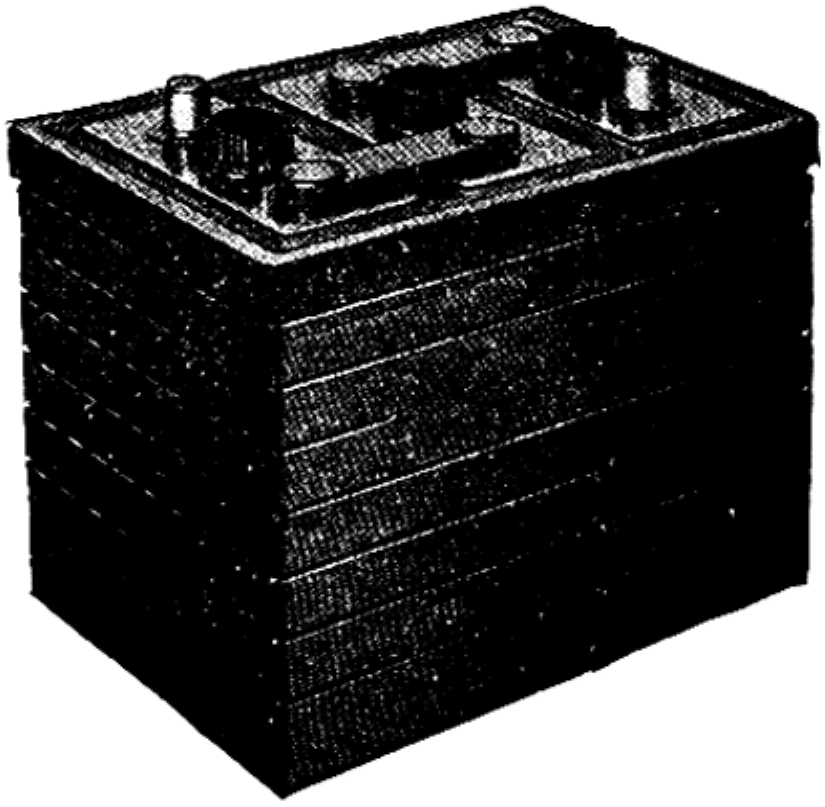
3. A line is called a one-dimensional space. This means that one measure, say 10 feet, is sufficient to tell how long any part of it is. It also means that the location of a point on a line can be given by one measure. Thus we can locate a car by saying that it is thirty miles from Harrisburg on the express highway between Harrisburg and Pittsburgh.

A plane, such as the surface of a flat table, is called a two-dimensional space. Again, this means that two dimensions, say length and width, are needed to tell the size of a part of a plane. It also takes two measures to locate a point in a plane. You learned this in your graph work in algebra. To locate a point, you used a horizontal or x distance and a vertical or y distance.

Objects that we can actually touch and carry cannot be contained in a plane. They occupy portions of space called geometric solids. A geometric solid is a portion of three-dimensional space. We shall refer to three-dimensional space simply as space. It takes three measures like length, width, and height to tell the size of any part of this space. To locate a point in space, it also takes three measures.

An airplane, for example, may be located by giving its longitude, latitude, and height above sea level.

Three-dimensional figures cannot be drawn in their true shape on a two-dimensional surface like a sheet of paper or the blackboard. A camera records on a plane the impacts of the rays of light reflected from a solid. As we view the picture we see a two-dimensional representation of a solid. This is also what happens on the retina of the human eye when we see a scene in real life. The human eye acts as a camera. Because of the experience that we have all had in viewing three-dimensional objects, we can visualize the flat picture recorded by the camera. Our problem will be to represent solid figures on a plane, approximating the way a camera does it, and to visualize such representations.

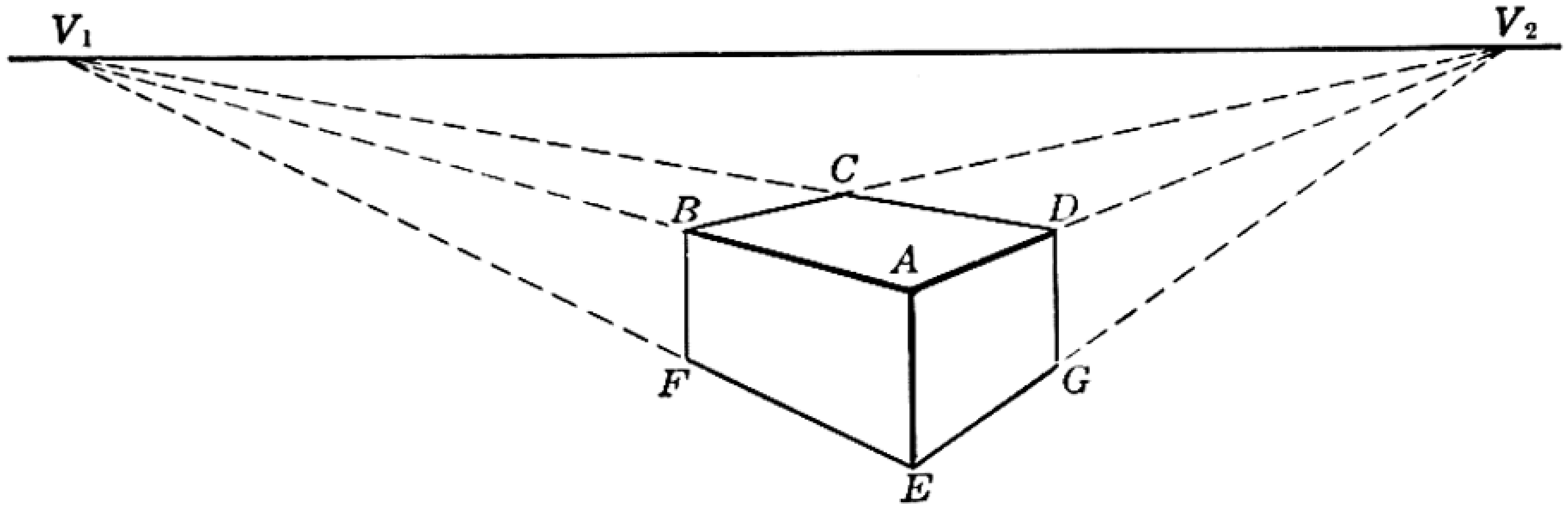


Suppose we wish to draw a chalk box on paper. We can draw only one face of the box in its true shape, a rectangle. The edges of this rectangle would all be parallel to our "picture plane," but we would see no other part of the box. We usually wish to represent as much of the solid as we can. To do this, we place ourselves to one side of the box and a little above (or below) it. Now we can see three faces. By drawing these faces as they would appear on the picture plate of a camera, we make a *perspective* drawing of the box. To make such drawings we use an important principle of perspective. It is the fact that certain lines that are parallel in three-dimensional space converge when represented on a plane.

We know that railroad tracks are everywhere equally distant and therefore parallel. As we look down along such tracks, however, the common distance between them appears smaller as it recedes from the eye. This causes the tracks to appear to converge, or come together. The point of convergence, called the *vanishing point*, appears to be the place where the picture fades out — on the eye level. This leads us to the first principle of perspective.

4. Parallel lines that recede converge at a vanishing point on the eye level.

5. To make an acceptable perspective drawing of our chalk box, we proceed as follows: Draw the vertical edge AE that is nearest to the eye. Draw the horizon, or the eye level, a little above the box



and select two vanishing points V_1 and V_2 on the horizon. Draw AV_1, the top edge of the box, and terminate it at B; draw the top edge AV_2 and terminate it at D. Note that AD is foreshortened more in proportion to its real length than AB because the vanishing point V_2 is closer than V_1. Draw EV_1, DV_1, EV_2, and BV_2. The vertical edges through B and D will then determine the remainder of the drawing.

By consulting a book on perspective in your library, you will learn that there are other principles to be observed in making such drawings. For our purposes, however, you can get along with the simple method described above and the additional suggestions that you will find as you progress in your work. After you have had some practice, you will be able to draw a suitable representation of a solid without using the actual vanishing points — by just remembering the principle of convergence. Of course, you will have to use your judgment about the length of some lines. Your best guide will probably be the trial-and-error method. If your drawing does not look right, try again.

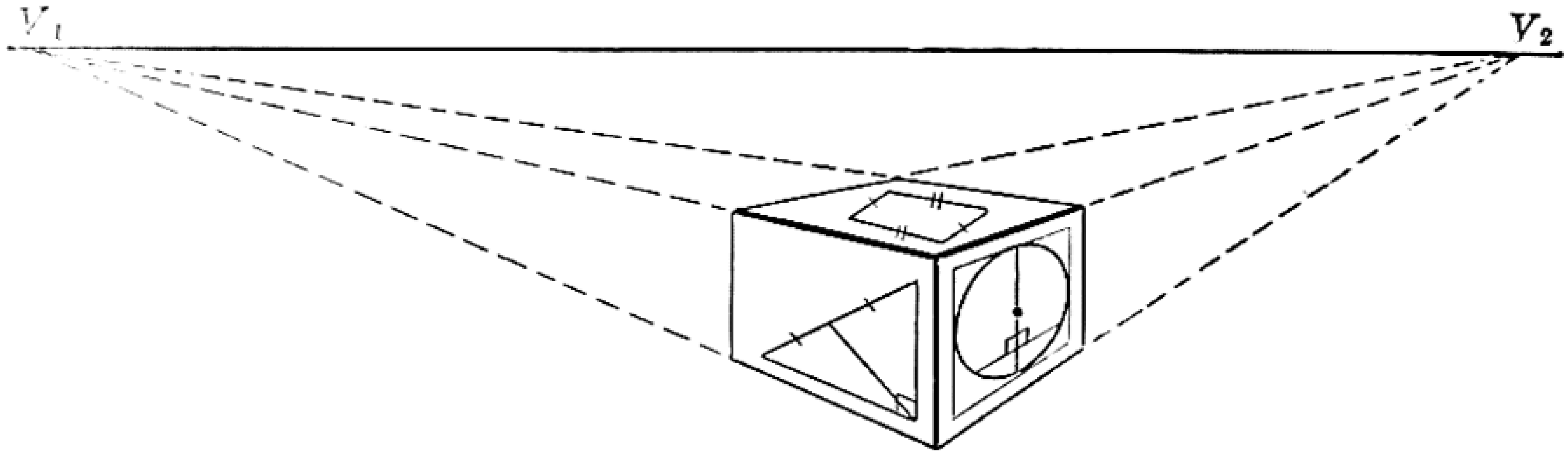
If our chalk box were a cube, the edge AE would represent its true length. The edge AB, although the same length as AE in space, would be shorter than AE in the drawing. The edge AD would be still shorter.

Draw a cube in perspective. Make the edges closest to you heavier and those farther away lighter.

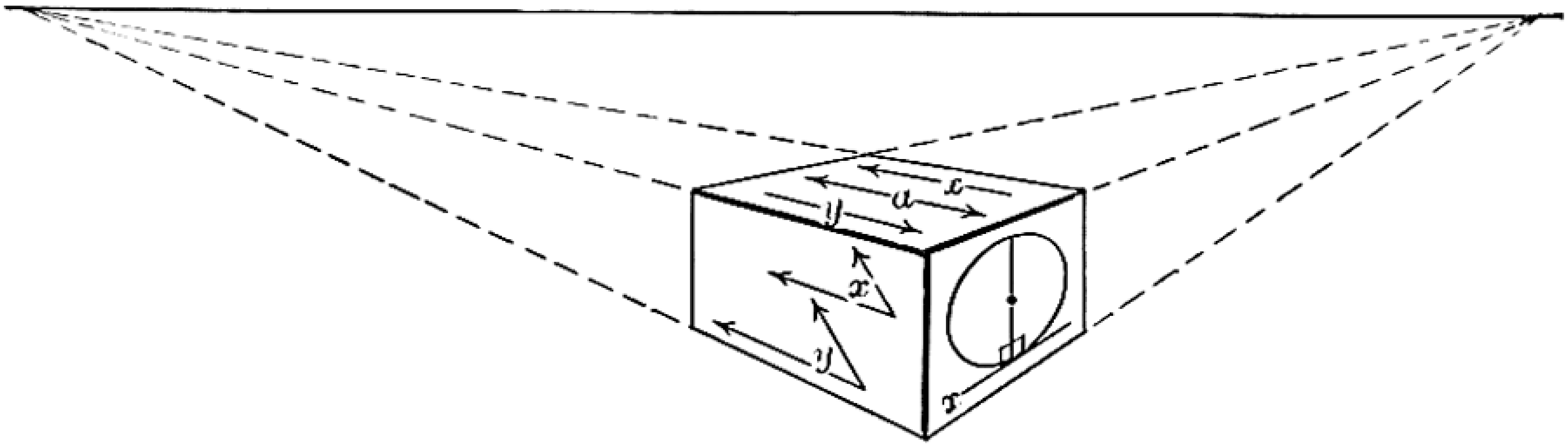
EXERCISES

In the following exercises you are to solve some problems in plane geometry. The figures illustrating these problems, however, are drawn in perspective on the faces of rectangular solids like a chalk box. Copy these figures carefully on a large scale and make whatever auxiliary constructions you need in your solution.

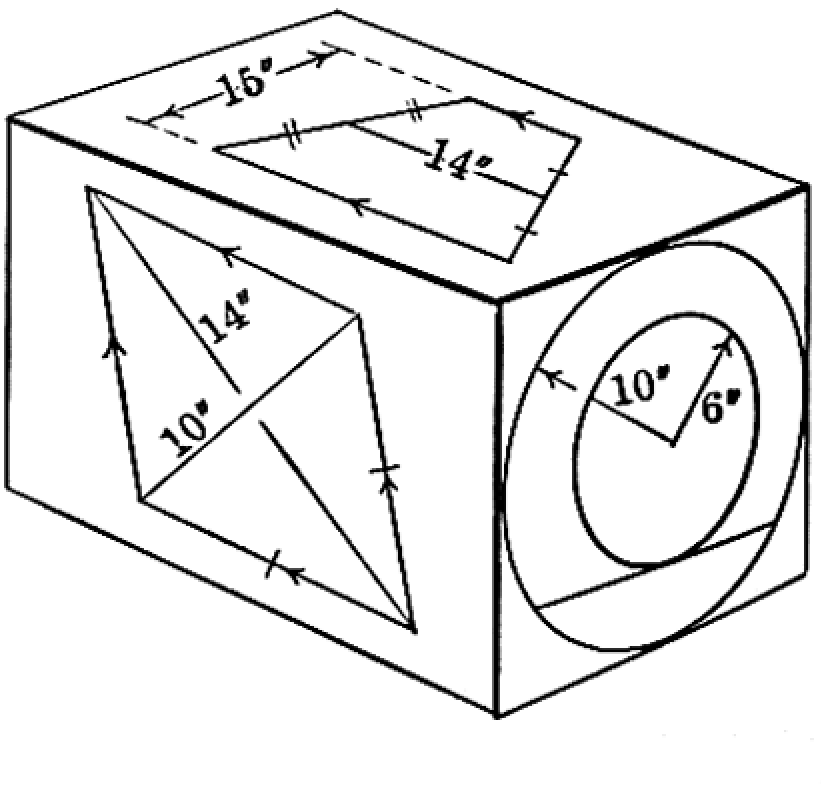
***Note:* The center of a rectangle in perspective is found by locating the intersection of its diagonals.**



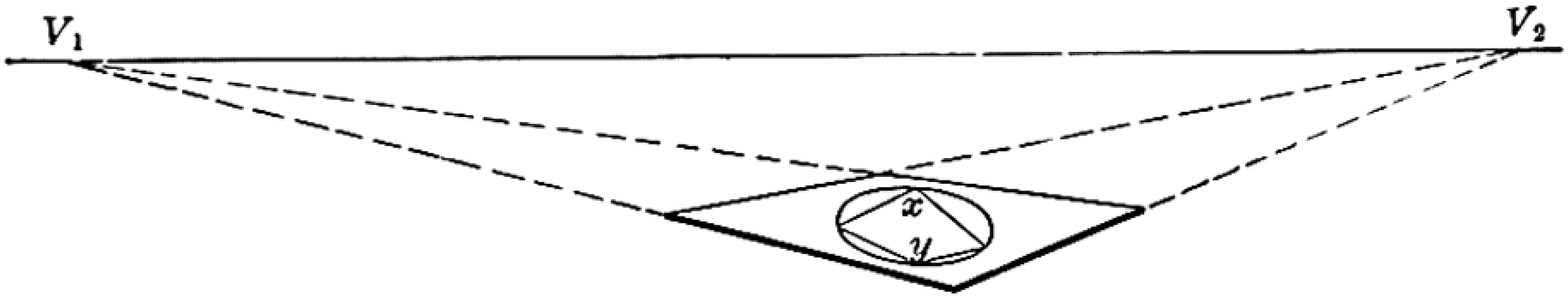
1. Prove: If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.
2. Prove: The median to the hypotenuse of a right triangle equals half the hypotenuse. (Complete the rectangle by extending the median its own length.)
3. Prove: A diameter perpendicular to a chord bisects the chord and its arcs.



4. Prove: If two lines are parallel to a third line, they are parallel to each other.
5. Prove: If the sides of one angle are parallel respectively to the sides of another angle and extend in the same directions, the angles are equal.
6. Prove: A line perpendicular to a diameter at one end is tangent to the circle. *Suggestion:* Assume that the line x intersects the circle and show that this would contradict Exercise 3.
7. The altitude of a trapezoid is 15 inches, and the line joining the midpoints of the legs is 14 inches. Find the area of the trapezoid.
8. The diagonals of a rhombus are 14 inches and 10 inches. Find the area of the rhombus.
9. The radii of two concentric circles are 10 inches and 6 inches. Find the length of a chord of the larger circle that is tangent to the smaller circle.

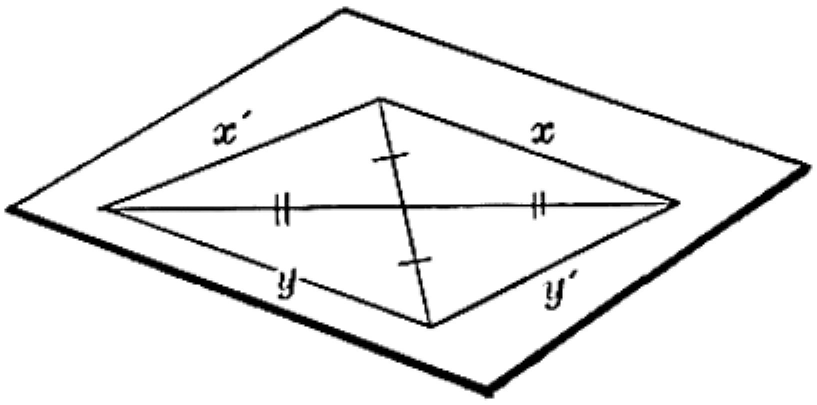


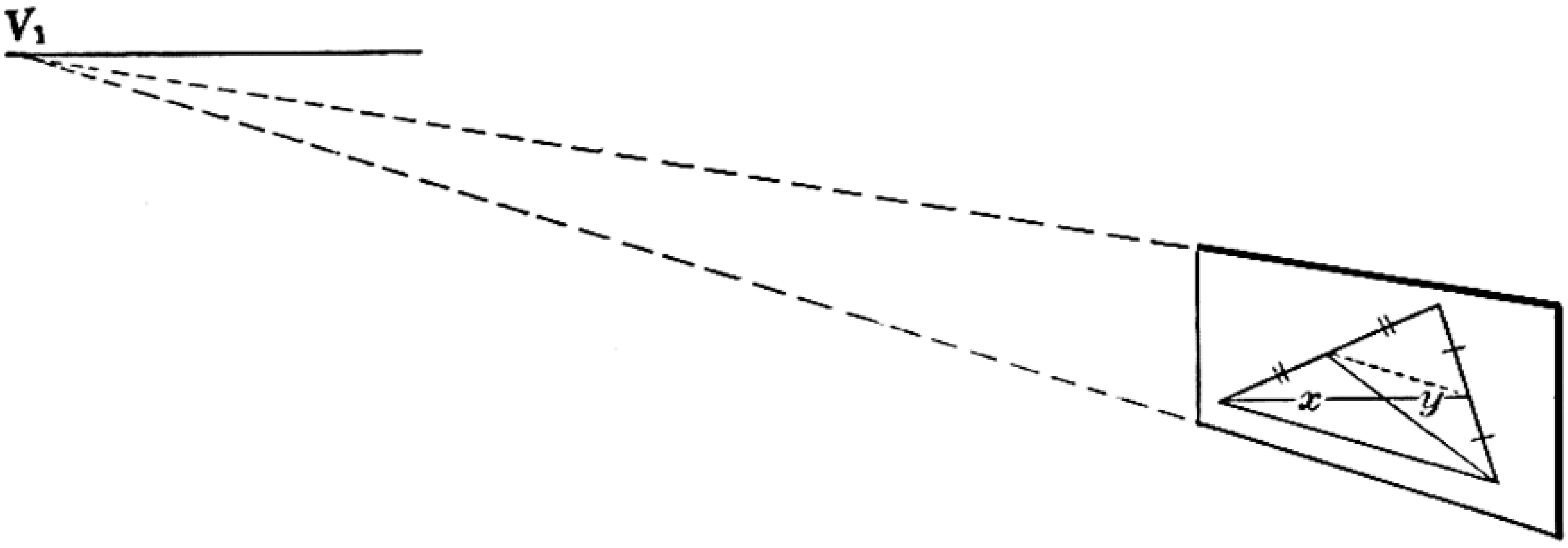
The diagrams that follow are also in perspective, but only one face of the rectangular solid is used. Because it is easy to draw, a parallelogram is sometimes used to represent a horizontal face *approximately*.



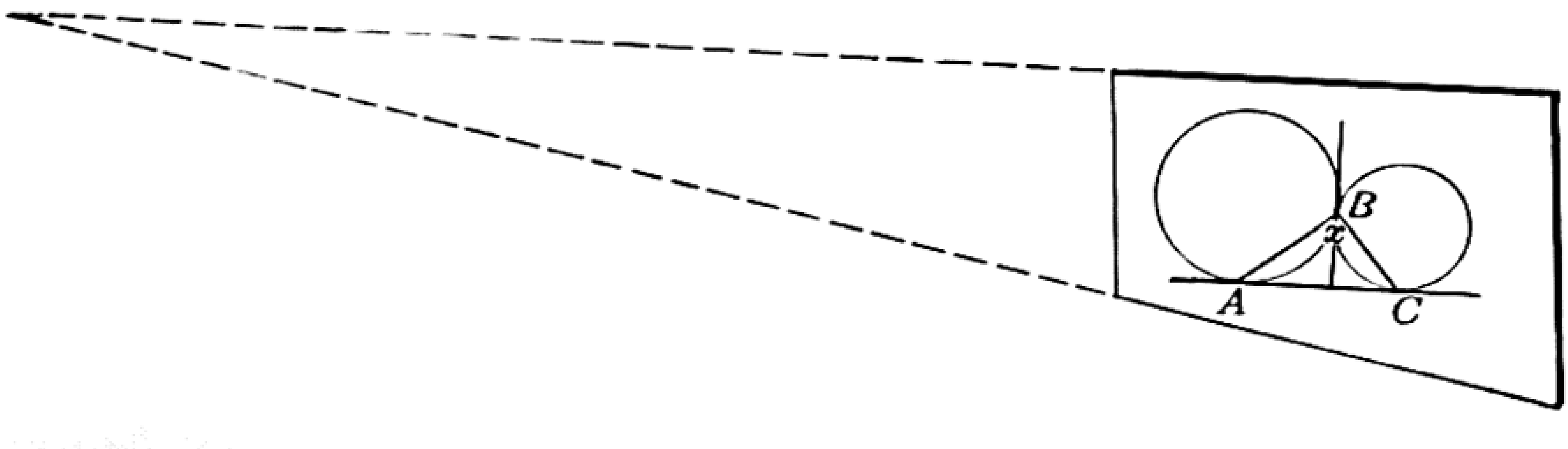
10. Prove: The opposite angles of an inscribed quadrilateral are supplementary.

11. Prove: If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.



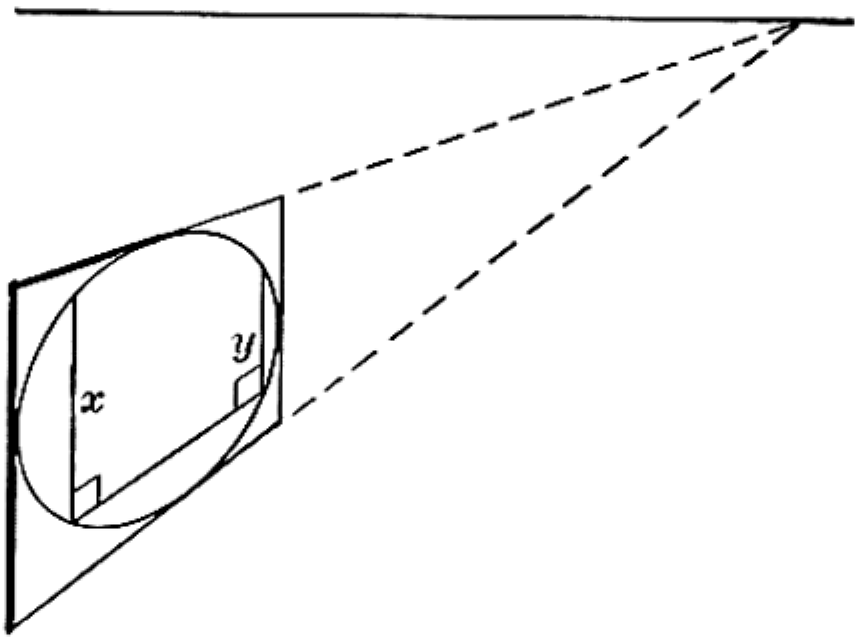


12. Prove: The medians of a triangle divide each other in the ratio of 2 to 1. *Suggestion:* Use similar triangles.

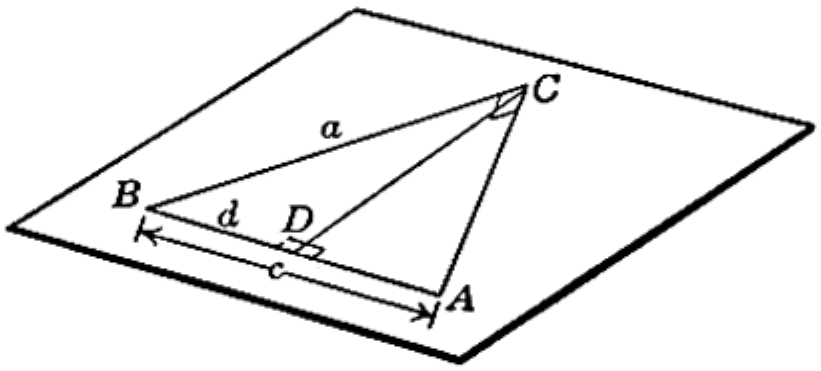


13. Prove: If two circles are tangent externally, the common external tangent and the chords joining the points of contact form a right triangle.

14. Prove: If two chords are perpendicular to a third chord at its ends, they are equal. *Suggestion:* Use Exercise 10.

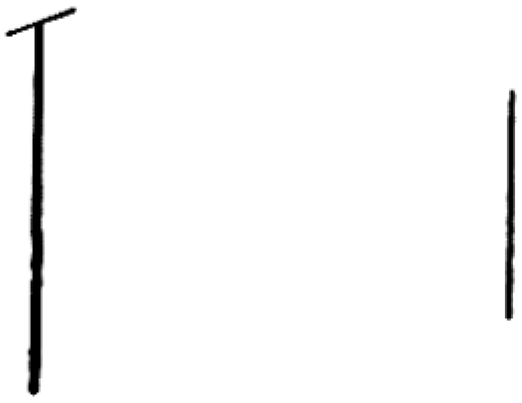


15. Prove: If the altitude is drawn to the hypotenuse of a right triangle, either leg is the mean proportional between the whole hypotenuse and the adjacent segment of the hypotenuse. *Suggestion:* Circumscribe a circle about $\triangle ACD$ and prove $a^2 = cd$.



SUPERIOR WORK

S 1. The figure represents the first and the fifth of a series of telegraph poles that are equally spaced along a straight road. The first pole has a crossarm for the wires.



- **a.** Copy this figure on a large sheet of paper.
- **b.** Using the intersection of the diagonals of the rectangles as centers, locate and draw the three intervening poles.
- **c.** Using the same principle as in (b), draw the sixth pole.
- **d.** Using the vanishing point, draw the crossbars on all the poles and three wires carried by them.

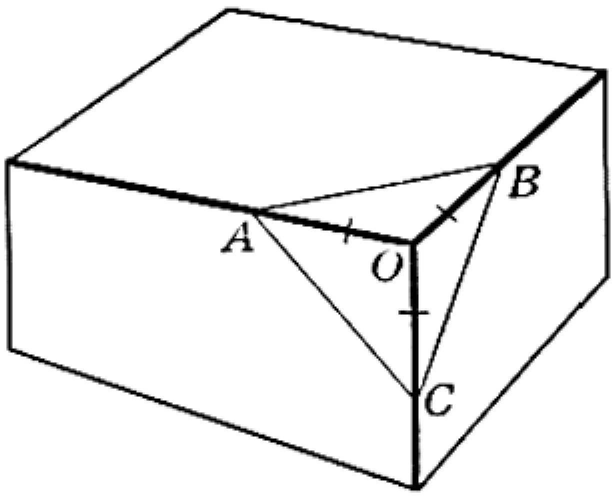
S 2. In your drawing for S 1, certain details like a picket fence, a sidewalk, thickness added to the poles and the crossarms, and shading would make it appear more real. Sketch some of these.

SPACE FIGURES

6. In the preceding lesson we were working with plane figures as seen in perspective. In our progress of emerging from the plane, we shall now work with figures that extend out of the plane. Some of these figures can be formed by folding plane figures so that they occupy more than one plane. Others are encountered in connection with rectangular solids that we have already learned to draw. Until we come to more formal definitions, we shall assume that a rectangular solid is a solid enclosed by six rectangles and that a cube is enclosed by six squares.

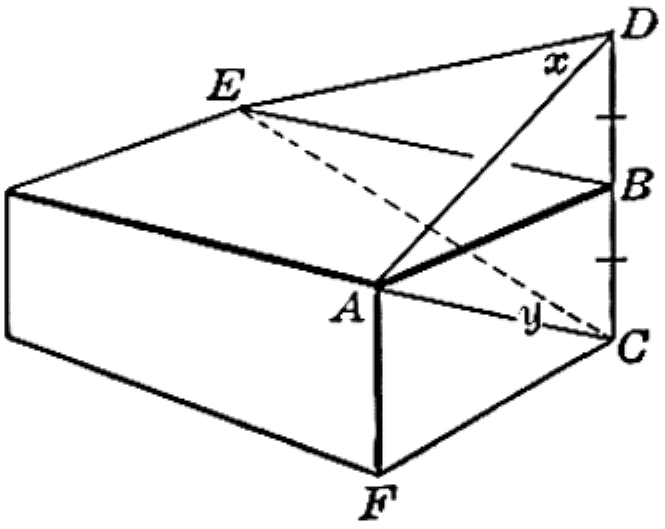
EXERCISES

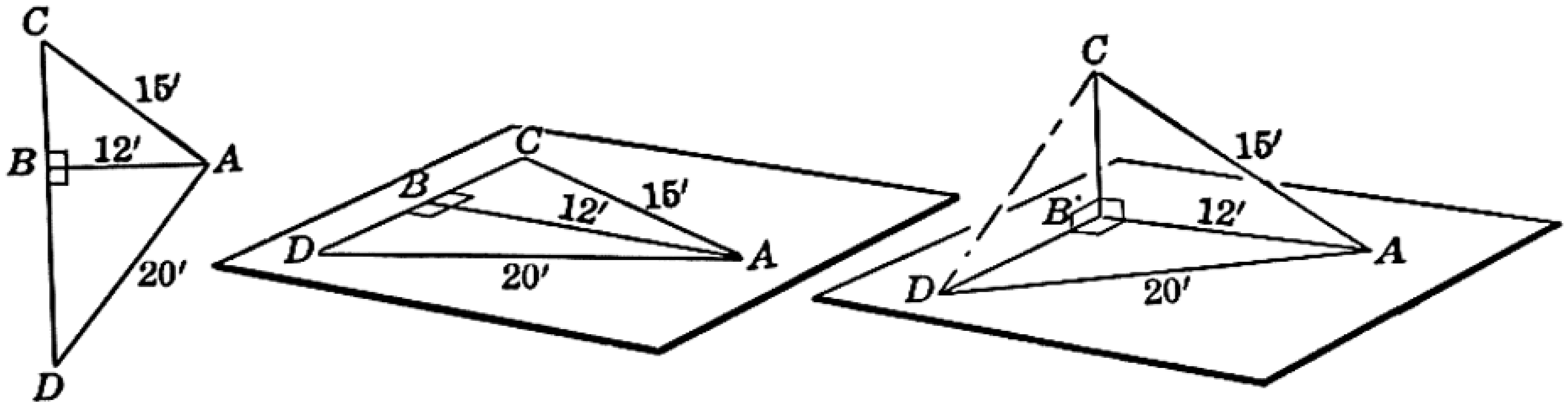
1. Three equal distances OA, OB, and OC are laid off on the edges of the rectangular solid shown in the figure. The points A, B, and C are then joined. Prove: $\angle ABC = 60°$.



2. In the same diagram let $OA = 5$ inches. Find the area of $\triangle ABC$.

3. The edge CB of the rectangular solid is extended its own length to D. The points C and D are then joined to the vertices A and E. Prove: $\angle ADE = \angle ACE$. The line CE is dotted to indicate that it is invisible from our point of view.



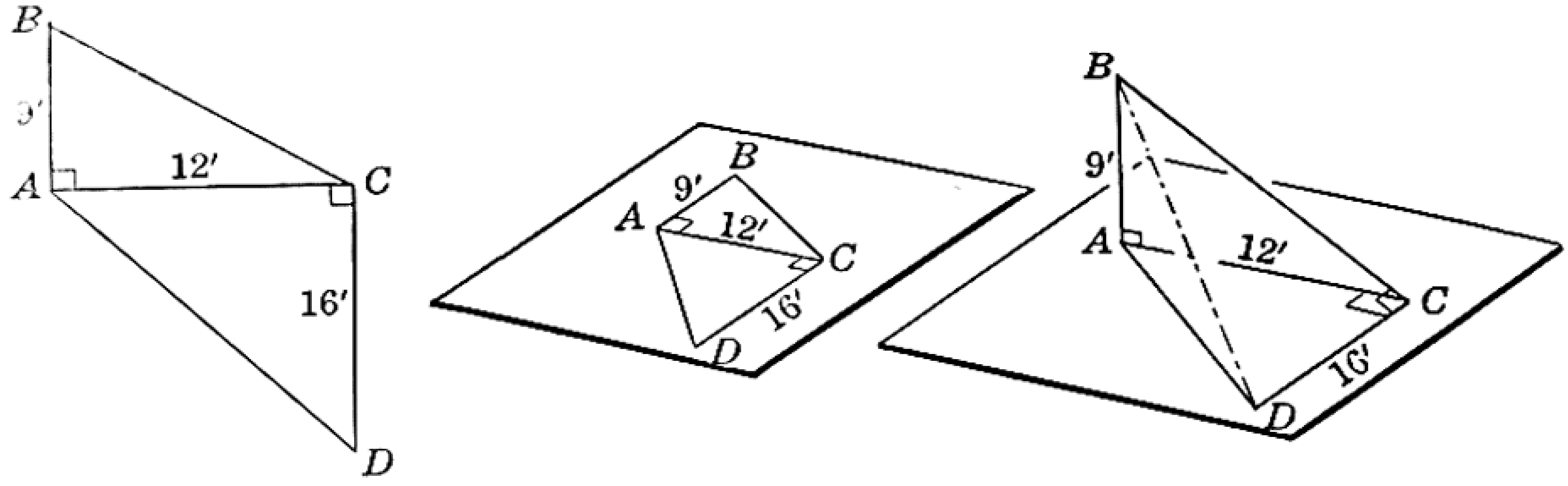


4. a. Using a scale of 5 feet to the inch, draw the plane figure shown on the left.

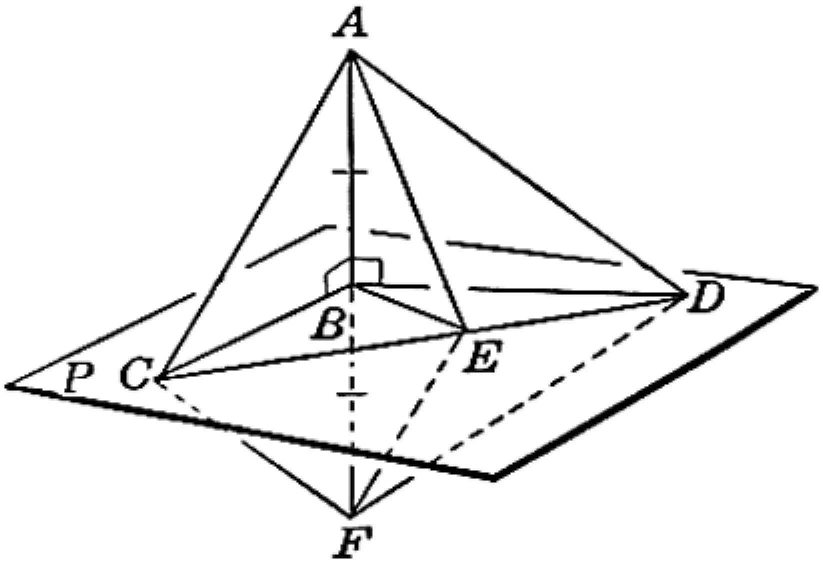
b. The second figure is the same as the first, but it is drawn in the perspective plane. Copy it.

c. Fold your first figure along AB until $\angle CBD$ has become 90°. Draw this figure in perspective, as shown at the right. Draw CD.

d. Find the length of the line CD in the space figure.



5. **a.** Using a scale of 5 feet to the inch, draw the plane figure shown on the left. We shall refer to this figure as a quadrilateral made up of two right triangles with a common leg.
   b. The second figure is the same as the first but it is drawn in the perspective plane. Copy it.
   c. Fold your first figure along the common leg AC until BC has become perpendicular to CD. Draw this in perspective, as shown at the right. Draw BD.
   d. Using the converse of the Pythagorean Theorem (§ 488), prove that BA has become perpendicular to AD.



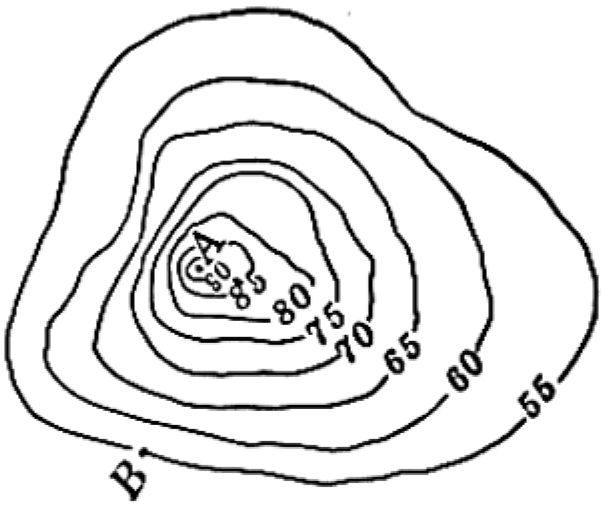
6. GIVEN: BC, BD, and BE in plane P.
   $AB \perp BC$ and BD at B.
   $AB = BF$.

   PROVE: $AB \perp BE$.

   Help: Prove $\triangle ACD \cong \triangle FCD$ and $\triangle ACE \cong \triangle FCE$.

SUPERIOR WORK

S 1. Each line on a *contour* map is the locus of points of equal elevation.
   a. Interpret contour lines close together; far apart.
   b. Can two contour lines cross each other? touch each other? Explain.
   c. At any point, what is the relation between the line of steepest slope and the contour line through that point?



Contour Map

S 2. **a.** Draw a contour map similar to the one in S 1. On your contour map sketch a line of "easy ascent" from B to A.

b. On your map draw an east-west line through A. From the points where this line intersects the contour lines draw perpendiculars to a horizontal line several inches below the map. Using a convenient scale for elevations, locate points at distances above the horizontal line to correspond to their contour lines. Sketch a curve through these points. Your curve is a profile along the east-west line.

c. Make a perspective drawing of your contour map.

PROPERTIES OF A PLANE

7. If a point moves, it generates the next higher space, which is a straight line or a curved line. If a straight line moves in a direction different from its own, it generates the next higher space, which is a plane surface or a curved surface. If a plane surface moves in a direction not contained within itself, it generates the next higher space, which is a three-dimensional space.

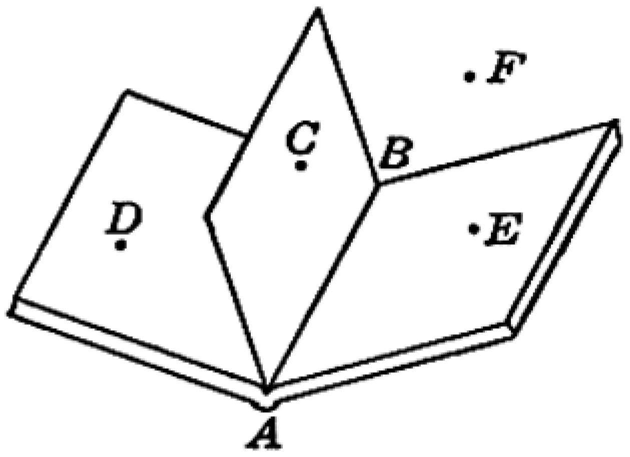
Another analogy that may help us to see the progression of the dimensions in space is the following: A point separates two parts of a line, a line separates two parts of a surface, and a surface separates space into two parts.

In plane geometry we considered points and straight lines as undefined terms. We shall now accept the term surface as undefined. In plane geometry we assumed that all our straight-line figures remained in the plane surface in which we were working. Thus, in the two-point postulate we assumed that two points determine a line. In addition, we took it for granted that such a line would forever remain in our plane. This "unwritten assumption" really defines a plane surface or a plane.

8. Definition. *A plane is a surface in which a line joining any two of its points lies entirely in the surface.*

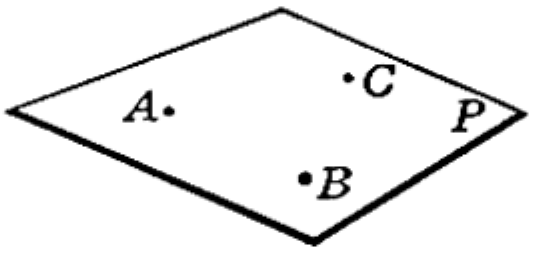
Determining a Plane

9. In plane geometry any number of lines can be drawn through a given point. Analogously, our experience with turning the pages of a book shows us that through a given line any number of planes may be passed. This experience also indicates that as we turn a page it will contain, in addition to the stationary edge AB, every other point in space — D, C, E, F, etc. — in only one position. This means that a straight line and an external point deter-



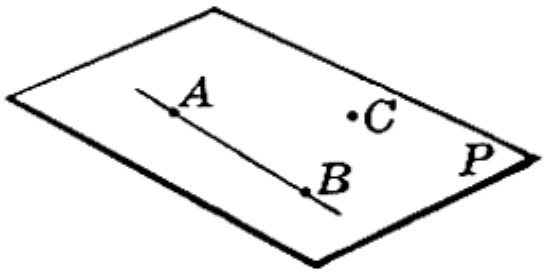
mine a plane. Since a straight line is itself determined by two points, we see further that three points not in one straight line determine a plane. Thus we assume the following three-point postulate in solid geometry that is analogous to the two-point postulate in plane geometry.

10. Three-Point Postulate. *Three non-collinear points determine a plane.*



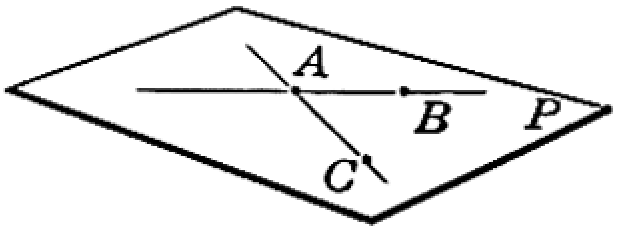
11. Corollary 1. A straight line and an external point determine a plane.

Any two points A and B on the given line and the external point C determine a plane. This plane must contain the given line by the definition of a plane.



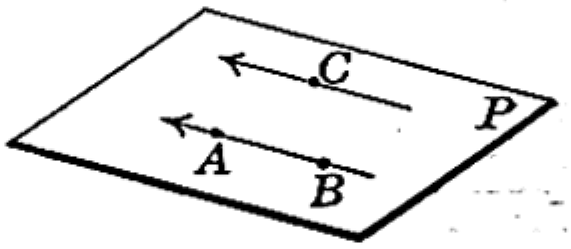
12. Corollary 2. Two intersecting lines determine a plane.

The point of intersection A and two other points B and C on each of the given lines determine a plane which contains the two given lines.



13. Corollary 3. Two parallel lines determine a plane.

By definition, parallel lines must be in one plane. This plane is determined by two points A and B on one line and a third point C on the other line.



Intersection of Space Figures

14. The intersection of two figures is the locus of points (§ 365) that are contained in both figures. In plane geometry we reasoned indirectly that two lines can intersect in only one point because, if they intersected in two points, it would contradict the two-point postulate. In solid geometry we reason that a line can intersect a plane in only one point because if it could intersect the plane in two points, then it would contradict the definition of a plane.

15. Corollary. The intersection of a line and a plane is a point.

16. Definition. *The point of intersection of a line and a plane is called the foot of the line.*

17. We would now like to show that, analogous to the intersection of two lines in a point, the intersection of two planes is a straight line. From the corollary that a straight line and an external point determine a plane, we can see that two planes can have no other points in common than those that are on one straight line. We can also see (from the definition of a plane) that if two planes do have two points in common, the line joining these points is in both planes. It does not follow from what we have assumed or proved so far, however, that two intersecting planes must have more than one point in common. Since we cannot prove our proposition from preceding propositions, we shall assume it.

18. Postulate. *The intersection of two planes is a straight line.*

EXERCISES

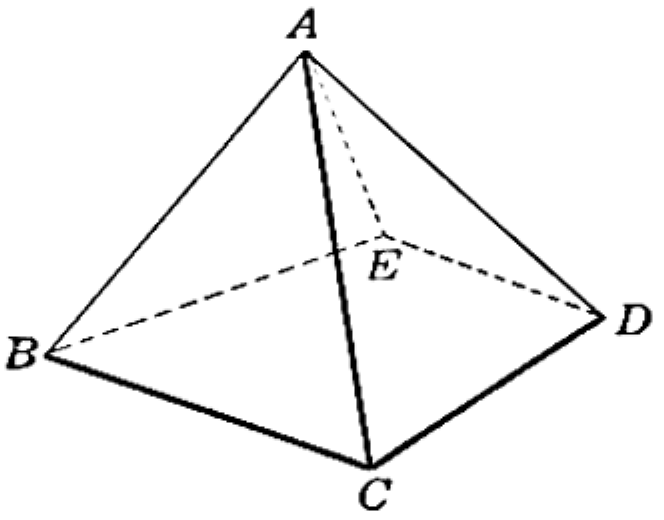
1. **a.** A girl argued that the definition of a plane (§ 8) is not reversible because two points on the surface of a cylinder may be joined by a straight line that will be entirely in the surface. Is her argument valid? Explain.
   b. Could the girl have used the same argument for the surface of a cone? for the surface of a sphere?

2. **a.** Support a book on the tips of your fingers. How many fingers must you use? What does this experiment illustrate?
   b. Why is a tripod used to support a camera or a transit?
   c. Why is a four-legged table sometimes unsteady?

3. **a.** What is the largest number of planes that can be determined by four points? five points? three parallel lines?
   b. Does an arc of a circle determine a plane? Why?

4. **a.** Why is a triangle a plane figure?
   b. Why does the median of a triangle lie in the same plane as the triangle?
   c. Why does a transversal of two parallel lines lie in the same plane as the parallel lines?

5. The figure is a pyramid.
    - a. How many planes are in the figure?
    - b. What is the greatest number of vertices in one plane? Name them.
    - c. Name two lines that are not in one plane.
    - d. If a plane were passed through AC and AE, the figure would be divided into two pyramids. How many planes (faces) would each pyramid have?



6. **a.** Is a quadrilateral always a plane figure? Why? (See definition of quadrilateral, § 390.)
    - **b.** Name some special quadrilaterals that are plane figures.
    - **c.** If the diagonals of a quadrilateral intersect, is the quadrilateral a plane figure? Why?
    - **d.** If two opposite sides of a quadrilateral intersect when extended, is the quadrilateral a plane figure? Why?

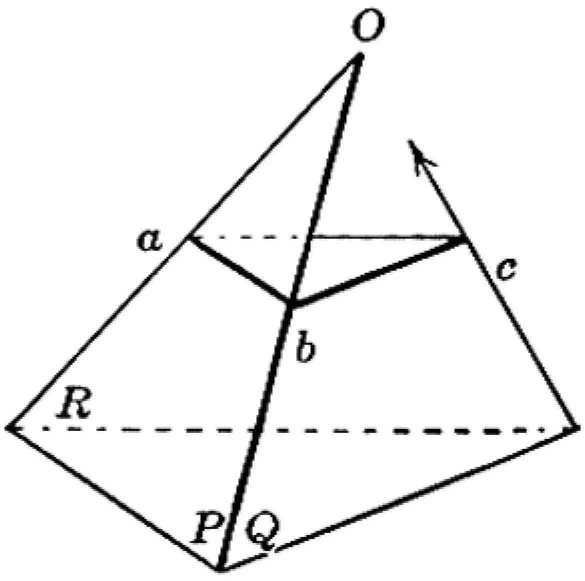
7. **a.** Are the diagonals of a parallelogram in one plane? of a quadrilateral? of a trapezoid?
    - **b.** Does a transversal of the sides of an angle lie in the same plane as the angle?
    - **c.** Under what conditions will two transversals of two lines lie in the same plane?

8. Prove that if three planes intersect and two of the intersections are not parallel, all three intersections meet in a point.

Given: In the figure the planes P, Q, and R intersect in a, b, and c. a and b meet in O.

Prove: O is on c.

Hint: Prove that O is in planes Q and R (§ 14).



SUPERIOR WORK

S 1. **a.** What is the largest number of lines that can be determined by n points?
    - **b.** What is the largest number of planes that can be determined by n points? by n concurrent lines?

S 2. Given: Lines l and m are not in the same plane, and neither one contains a given point X. Plane P, determined by l and X, cuts m in Y. Plane Q, determined by m and X, cuts l in Z.
Prove: X, Y, and Z are collinear.

S 3. Complete the following statements, using the words *necessary* and/or *sufficient.*

- **a.** For three points to determine a plane, it is —?— that they be on two different lines.
- **b.** For two lines to determine a plane, it is —?— that they intersect.
- **c.** For a line and a point to determine a plane, it is —?— that the point be outside the line.
- **d.** For a surface to be a plane, it is —?— that it contain two intersecting lines.
- **e.** For a surface to be a plane, it is —?— that any line in the surface when extended remain in the surface.
- **f.** For a surface to be a plane, it is —?— that no line intersect it in two points.
- **g.** For two lines to be parallel, it is —?— that they do not intersect.
- **h.** For two lines not to intersect, it is —?— that they be parallel.

ADAPTING PLANE GEOMETRY TO SOLID FIGURES

19. We have observed that solid geometry is an extension of plane geometry. In applying our knowledge of plane geometry to figures in space, however, we shall have to observe certain precautions and make certain modifications. For example, the condition that parallel lines must be coplanar (in the same plane) caused us no concern in plane geometry, where all our figures were in the same plane. In space, on the other hand, certain lines cannot be in one plane. Such lines are neither parallel nor do they meet. They are called *skew* lines.

20. Definition. *Skew lines are lines that cannot be contained in the same plane.*

In plane geometry we could begin an indirect proof with the statement, "Either the lines x and y are parallel or they intersect." In three-dimensional space such a statement would not be valid because there is a third possibility — the lines may be neither.

21. It is also possible to have a skew quadrilateral, that is, a quadrilateral not all in one plane (as you have seen in the preceding lesson). In general, the word *skew* is opposite to *coplanar*. We shall find that (1) certain propositions that we have proved in plane geometry are not valid in space at all, and (2) certain propositions that are true in space will have to be proved again because the proof in plane geometry was not general enough.

If a figure can be contained in one plane, then any proposition that we have proved about it in plane geometry may be used in solid geometry. If the figure cannot be contained in one plane, then such a proposition may be false altogether. Even if it is valid, we may not use it in formal demonstration until we have given a more general proof of it — a proof that applies in three-dimensional space.

Constructions in Space

22. Another important difference between plane and solid geometry is in the matter of constructions. For example, in plane geometry we can use a straightedge to actually construct a line through two given points as authorized by the two-point postulate. In three-dimensional space, on the other hand, we cannot follow the three-point postulate and actually construct a plane through three non-collinear points. We can draw only a representation of a plane and, through the principles of perspective, make it appear as if it were drawn through the three given points.

We must remember that the constructions we learned in plane geometry can be performed only in a plane. Thus, if we wish to construct a perpendicular to a given line through an external point, we must first draw the plane that is determined by the given line and the given external point. In that plane, then, we draw a representation of the required perpendicular. Note that we cannot construct this perpendicular with compass and straightedge because the plane is usually in perspective. If the given point were on the given line, we would proceed in the same way. This time, however, the plane would not be determined. Consequently we could draw an indefinite number of perpendiculars through the given point — one in each plane containing the given line.

23. THEOREM. Through a given point on a given line any number of perpendiculars can be drawn to the given line; through a given point not on a given line one perpendicular, and only one, can be drawn to that line.

In solid geometry construction problems will be in the form of theorems. We will thus *prove* what constructions are *possible* in terms of constructions in plane geometry.

EXERCISES

On pages 2–4 you answered certain questions as they pertain to figures in a plane. Reconsider some of these questions now so that your answers will apply to figures in space. Some of your answers may be intuitive (§ 2) for the present.

1. Page 2, Exercise 1.

2. Page 2, Exercise 2.

3. Page 2, Exercise 4.

Complete Exercise 4 (a) by writing: "Not necessarily parallel."

4. Page 2, Exercise 5.

5. Page 3, Exercise 6.

6. Page 3, Exercise 9.

7. Which of the six definitions in Exercise 13 on page 4 must be modified in space?

8. Considering figures in space, which of the propositions that you have proved in Exercises 1–6 on page 8 are valid without further proof? Which need more general proof? Which are not true?

9. Describe how to draw a line parallel to a given line through a given point in space.

10. **a.** Describe how to draw a line perpendicular to one of two given skew lines so that it will intersect the other.

b. Is more than one such perpendicular possible? If so, are they all in one plane?

SUPERIOR WORK

S 1. The opposite of a proposition is formed by changing the hypothesis and the conclusion of the proposition to their opposites (§ 362).

For example, the following two propositions are opposites of each other:

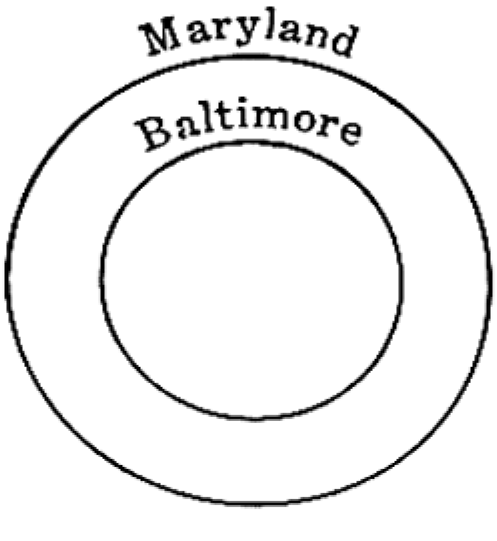
If two lines are skew, they are not parallel.
If two lines are coplanar, they are parallel.

Consider the following propositions.

A: If a building is in Baltimore, it is in Maryland.
B: If a building is not in Baltimore, it is not in Maryland.
C: If a building is not in Maryland, it is not in Baltimore.

Proposition B is the opposite of proposition A, and proposition C is the converse of B. C is therefore the converse of the opposite or, more briefly, the opposite-converse of A. The opposite-converse is sometimes called the *contrapositive*.

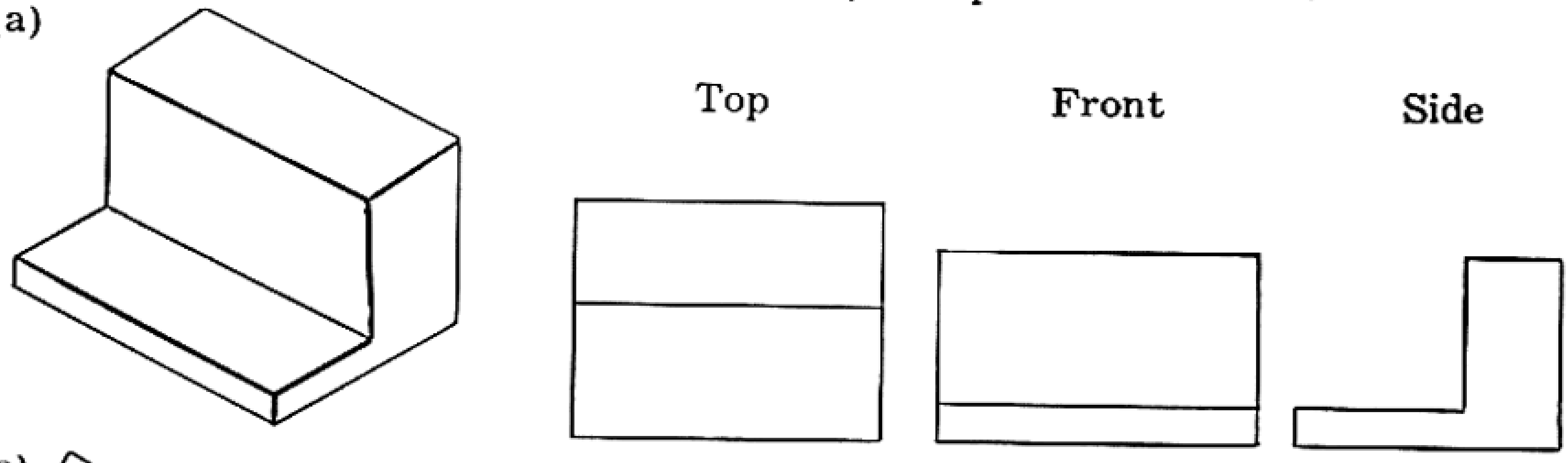
The figure used on page 4 to illustrate proposition A also illustrates proposition C; it shows that every building outside Maryland is outside Baltimore. The figure illustrates that a proposition and its opposite-converse are equally valid (§ 436) and that proposition B, the opposite of A, is not valid, for a building not in Baltimore may still be in Maryland.



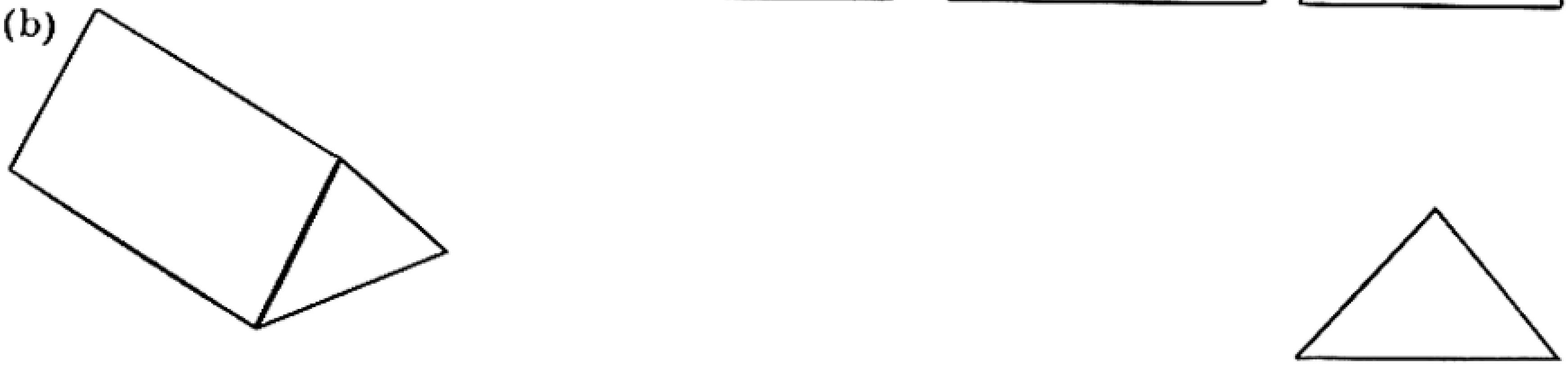
Form the opposite-converse of each proposition in Exercises 4–9 on pages 2–3. Tell if they are valid in plane and in solid geometry.

S 2. In each row below, a perspective drawing is to be followed by the top, front, and side views in their true shapes. Copy the completed drawings and, using (*a*) as a model, complete the others.

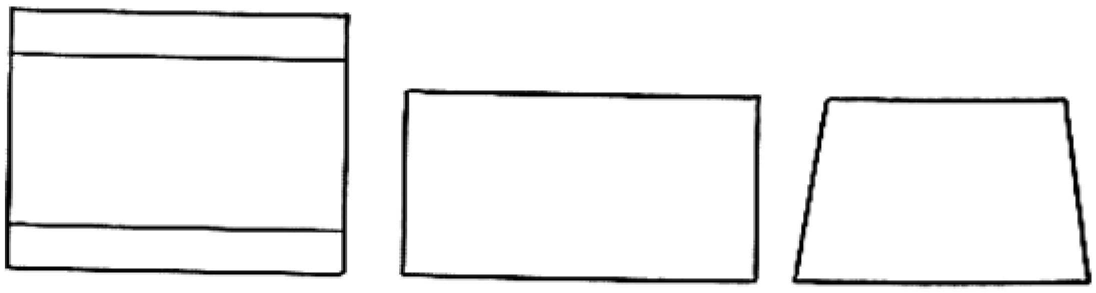
(a)



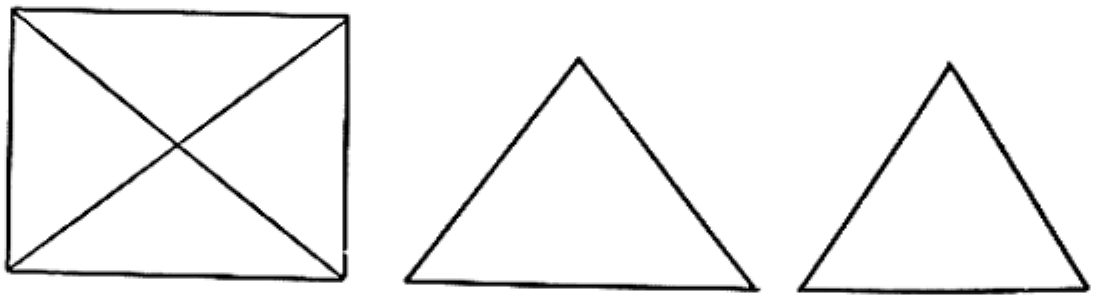
(b)



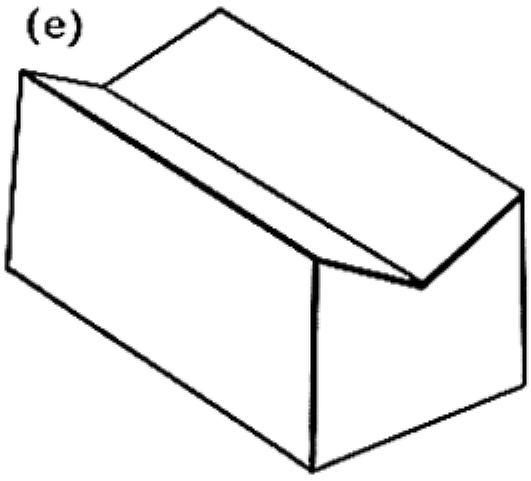
(c)

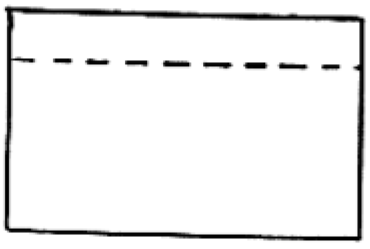


(d)



(e)



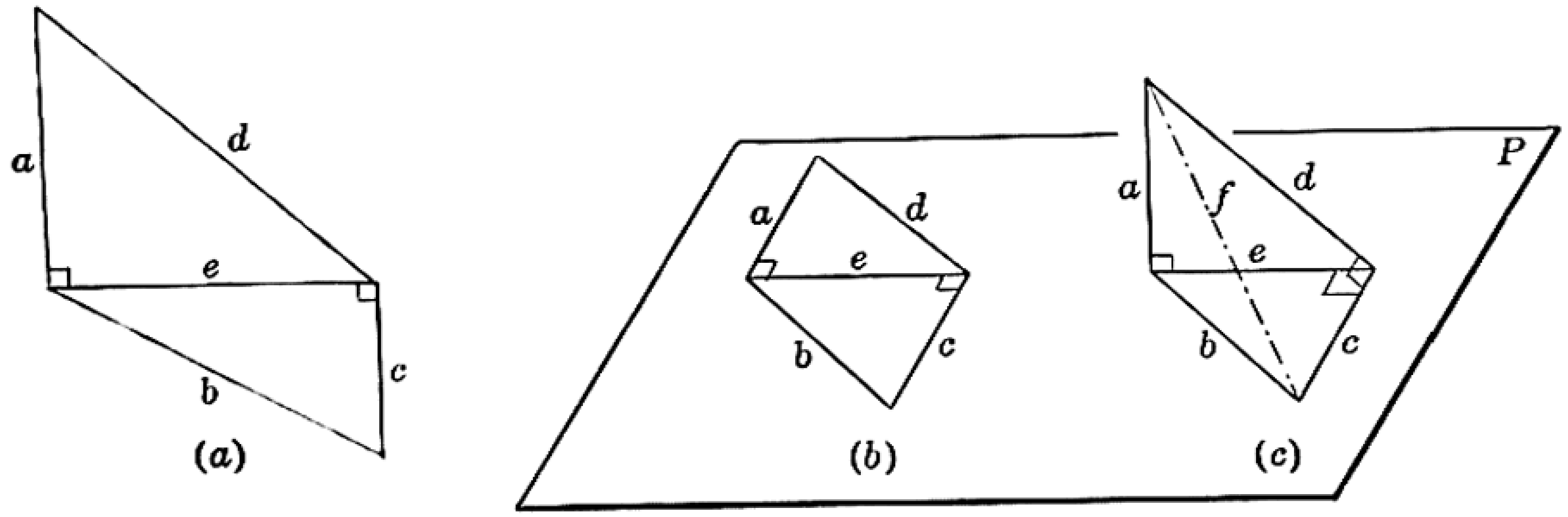


A Line Intersecting a Plane

A PERPENDICULAR TO A PLANE

24. A simple way to emerge from a plane is to fold a plane figure and then study the new relationships that arise in the space figure. We shall do this with a quadrilateral made up of two right triangles with a common leg (page 12, Exercise 5). The following theorem will prove that, when this figure is folded along the common leg, a space figure with four pairs of perpendicular lines can result. The figure is drawn in three ways: (*a*) as it appears in its true shape in plane geometry, (*b*) as a plane figure in perspective, and (*c*) as a figure in space after folding.

25. THEOREM. If a plane quadrilateral made up of two right triangles with a common leg is folded along the common leg until two of its sides become perpendicular, then the other two sides become perpendicular. (Skew Quadrilateral Theorem.)



(*a*) (*b*) (*c*)

Given: In Fig. (*c*), skew quad. *abcd* in which diagonal $e \perp a$ and c, and $d \perp c$.

Prove: $a \perp b$.

Cons.: Draw space diagonal f.

Proof: *Statements*		*Reasons*
1. From rt. $\triangle ade$, From rt. $\triangle bce$,	$d^2 = a^2 + e^2$. $c^2 = b^2 - e^2$.	1. Pythagorean Theorem.
2.	$d^2 + c^2 = a^2 + b^2$.	2. Add. Ax.
3. Since $d \perp c$,	$d^2 + c^2 = f^2$.	3. Pythagorean Theorem.
4. Hence,	$a^2 + b^2 = f^2$,	4. Subs. Ax.
5. and	$a \perp b$.	5. Converse of Pythagorean Theorem (§ 488).

Prove that if $a \perp b$, then $d \perp c$.

26. The theorem about the skew quadrilateral will help us understand the idea of a perpendicular to a plane and to prove certain propositions about it. Thus, in the figure for this theorem we may imagine that b rotates in the plane P about the foot of a so that we have a quadrilateral with a rotating side. When d is perpendicular to c, a will be perpendicular to all positions of b. This would *indicate* that at any point in a plane there exists a line that is perpendicular to all the lines in the plane passing through its foot. We call such a line a perpendicular to the plane.

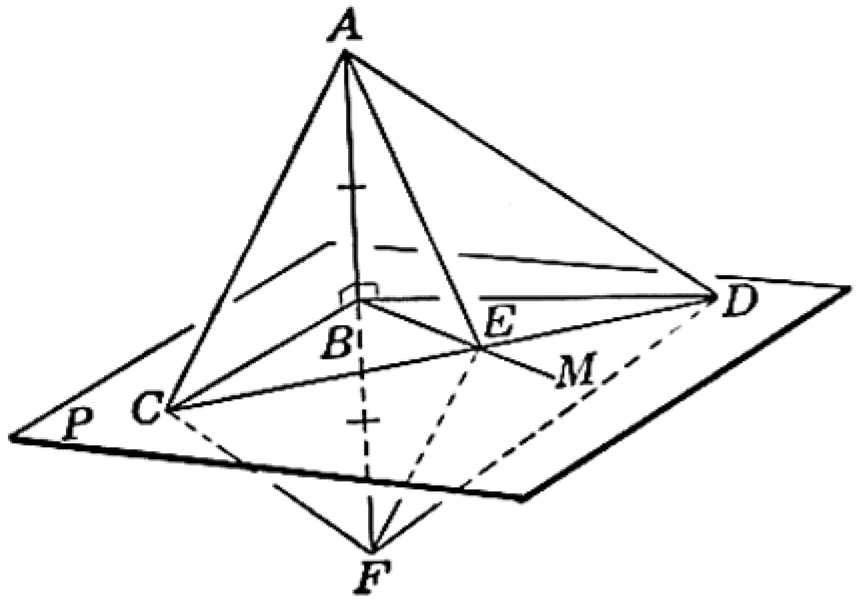
27. Definition. *A perpendicular to a plane is a line that is perpendicular to every line in the plane passing through its foot.*

28. If a line is perpendicular to a plane, the plane is said to be perpendicular to the line. The term perpendicular when used as a noun refers to a line rather than a plane.

29. To prove that a line is perpendicular to every line in a plane passing through its foot would be an endless task. The line would be perpendicular to the plane if we could prove that it is perpendicular to one line in the plane, provided that this line is a *general* line and not any one in particular. The line would have to be so general that it could represent all the lines of its kind. The implication is that if the proof holds for this line, it could be repeated for all the lines that it typifies. In the following theorem we shall use this principle to prove that if a line is perpendicular to two given lines in another plane, then it is perpendicular to any line whatsoever in that plane and passing through the intersection of the other three lines. We shall then have two ways of proving a line perpendicular to a plane: (1) by proving that it is perpendicular to a representative line in the plane passing through its foot, and (2) by proving that it is perpendicular to two given lines in the plane through its foot.

Analysis. In plane geometry you learned that analysis is an important method for discovering a proof. Study this analysis of the next theorem before reading the synthetic proof: To prove that AB is perpendicular to the plane P, we must show that it is perpendicular to a *general* line such as BM in the plane. From plane geometry we recall that the two-point theorem (§ 449) can be used to prove two lines perpendicular. This suggests that we extend AB to F so that B will be equidistant from A and F. Now we need only prove that some other point E on BM is also equidistant from A and F. This we can do by drawing the auxiliary lines and using congruent triangles. You may have already done this in Exercise 6 on page 12.

30. THEOREM. If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane of these lines.



GIVEN: $AB \perp$ intersecting lines BC and BD in plane P.

PROVE: $AB \perp P$.

CONS.: Through B draw *any* line BM in P.

Draw a line cutting BC, BD, and BM in C, D, and E respectively.

Extend AB to F, making $BF = AB$.

Draw AC, AD, AE, FC, FD, and FE.

PROOF:

Statements	*Reasons*
1. $AC = CF$. $AD = DF$.	1. Every pt. on the $\perp$ bisector of a line segment is equidistant from the ends of the line segment. (§ 450)
2. $CD = CD$.	2. Identity.
3. $\triangle ACD \cong \triangle FCD$.	3. s.s.s.
4. $\angle ACE = \angle FCE$.	4. Def. $\cong$ $\triangle$s.
5. $CE = CE$.	5. Identity.
6. $\triangle ACE \cong \triangle FCE$.	6. s.a.s.
7. $AE = EF$.	7. Def. $\cong$ $\triangle$s.
8. $AB \perp BM$.	8. Two-pt. Theorem. (§ 449)
9. $AB \perp P$.	9. Def. $\perp$ to plane. (§ 27)

You can readily see that there are more relationships in solid geometry than in plane geometry. To get a clearer idea of these relationships, group your propositions into larger units. In this lesson we are concerned entirely with a perpendicular to a plane. From the skew quadrilateral theorem (§ 25) we saw what such a perpendicular means. We also have two ways of proving a line perpendicular to a plane, namely, the definition of a perpendicular to a plane and the preceding theorem. The remaining propositions in this lesson will prove that through a given point a plane, and only one plane, can be drawn perpendicular to a given line; and through a given point a line, and only one line, can be drawn perpendicular to a given plane. Before studying these propositions, read again § 22.

31. THEOREM. Through a given point a plane can be drawn perpendicular to a given line.

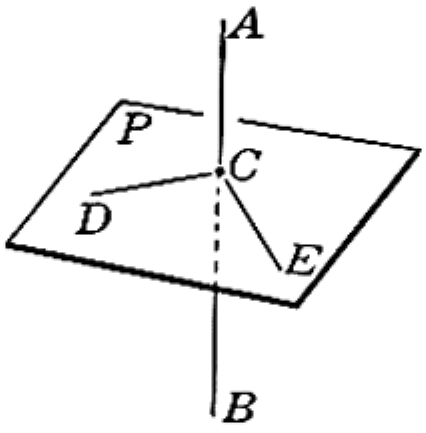


Fig. 1

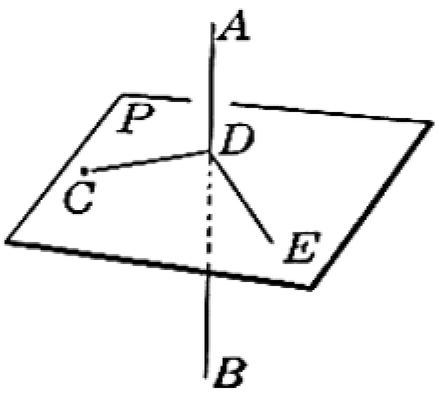


Fig. 2

Given: AB and pt. C.

Prove: A plane can be drawn through $C \perp AB$.

Proof:

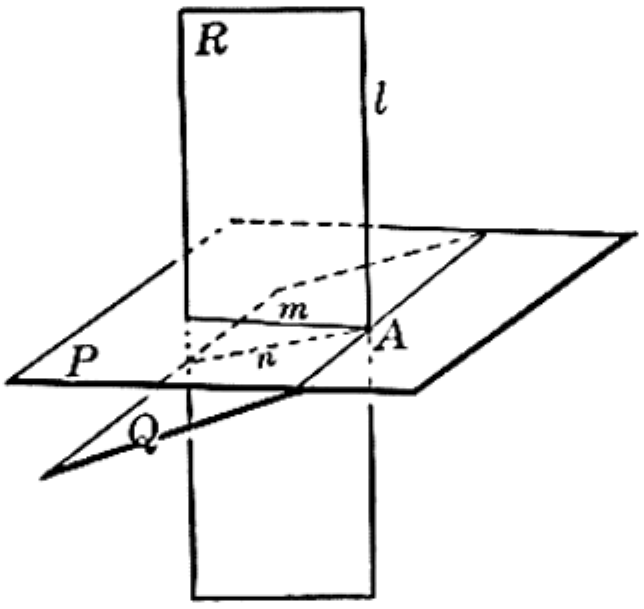
Statements	*Reasons*
1. Through C draw $CD \perp AB$.	1. Through a given pt. a $\perp$ can be drawn to a given line. (§ 23)
2. Draw CE (DE) $\perp AB$.	2. Same as 1. (The parentheses refer to Fig. 2.)
3. Pass plane P through CD and CE (DE).	3. Two intersecting lines determine a plane. (§ 12)
4. $P \perp AB$.	4. A line $\perp$ to 2 intersecting lines at pt. of intersection is $\perp$ to plane of lines. (§ 30)

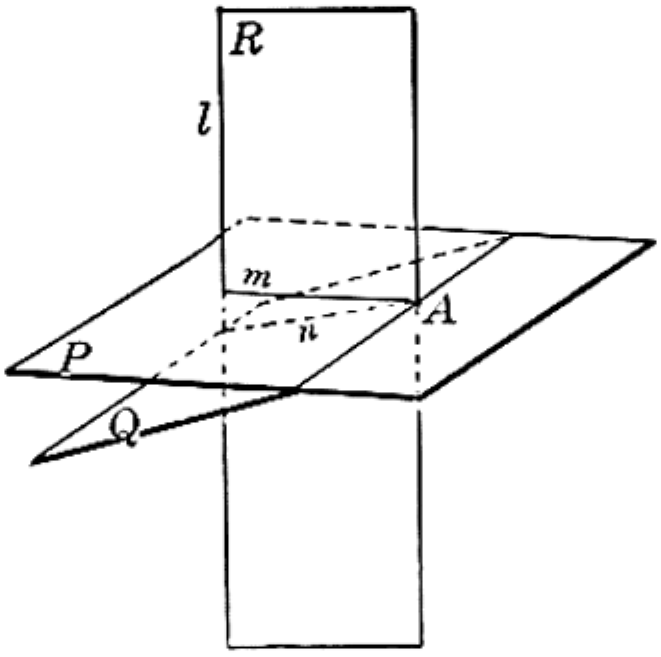
Note once again that we did not actually construct plane P with compass and straightedge as in plane geometry. What we did was to use the earlier propositions § 23, § 12, and § 30 to show that the plane P *exists*. We will next show that the plane P is *unique*.

32. To prove that through a given point only one plane can be perpendicular to a given line, we use the indirect method. This method will be used more often in solid geometry than in plane geometry. We therefore recall from plane geometry the essentials of this type of proof.

In the opening statement of the proof we list all the possibilities. We then show that all the possibilities except one contradict some proposition that we have previously accepted. Finally we conclude that the remaining possibility must be accepted. In the following proof we shall assume that two planes through the same point are both perpendicular to the same line. We shall then show that this would result in two lines through the same point being perpendicular to a third line, all in the same plane. This of course contradicts a proposition in plane geometry.

33. THEOREM. Through a given point only one plane can be perpendicular to a given line.





GIVEN: Line l and point A (on l or outside). Plane P contains A and is $\perp l$.

PROVE: No other plane through A is $\perp l$.

INDIRECT PROOF:

Statements	*Reasons*
1. Either there is another plane through $A \perp l$ or there is not.	
2. Assume that plane Q also passes through A and is $\perp l$.	
3. Pass plane R through l and A.	3. A plane may be drawn through a line and a pt. (§ 11)
4. R will intersect P and Q in lines m and n respectively.	4. The intersection of 2 planes is a st. line. (§ 18)
5. $m \perp l$, and $n \perp l$.	5. Def. $\perp$ to plane. (§ 27)
6. This is impossible.	6. In one plane (plane R), only one $\perp$ can be drawn to a given line through a given pt. (§ 539)
7. No other plane through A is $\perp l$.	

The following corollary summarizes the theorem in § 30 about a line perpendicular to two intersecting lines and the theorem we have just proved about the uniqueness of a plane perpendicular to a given line at a given point on the line.

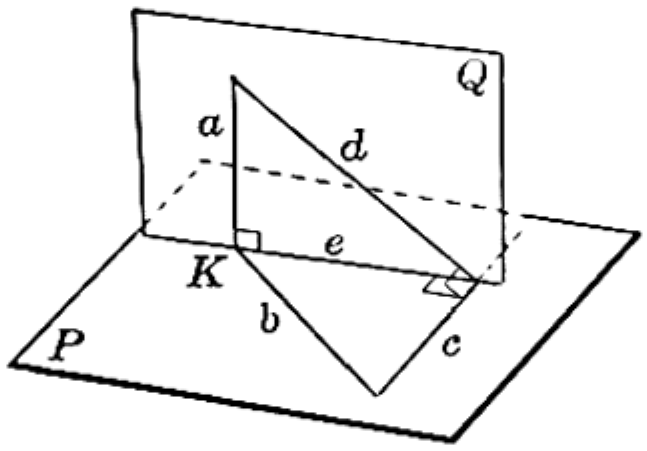
34. Corollary 1. Two lines perpendicular to the same line at the same point determine the plane perpendicular to the line at that point.

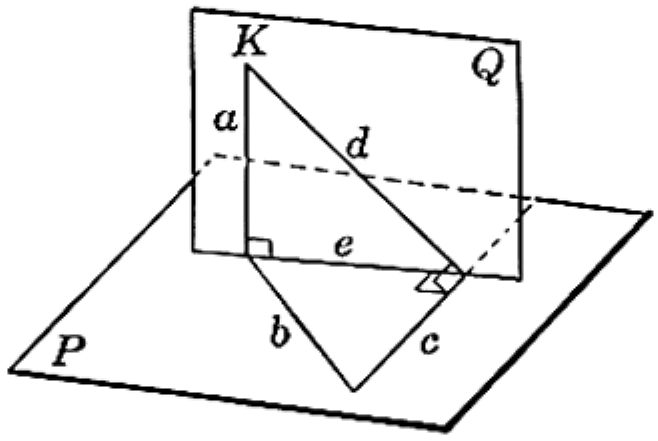
If more than two lines are perpendicular to the same line at the same point, any two of them will determine a plane perpendicular to the line at that point. Since there is only one such plane, all the perpendiculars must lie in it.

35. Corollary 2. All the perpendiculars that can be drawn to a given line at a given point on the line lie in a plane perpendicular to the given line at the given point.

We shall now see how the theorem about the skew quadrilateral (§ 25) enables us to construct a perpendicular to a given plane.

36. THEOREM. Through a given point a line can be drawn perpendicular to a given plane.





GIVEN: Pt. K in or outside plane P.

PROVE: A line can be drawn through $K \perp P$.

CONSTRUCTION:

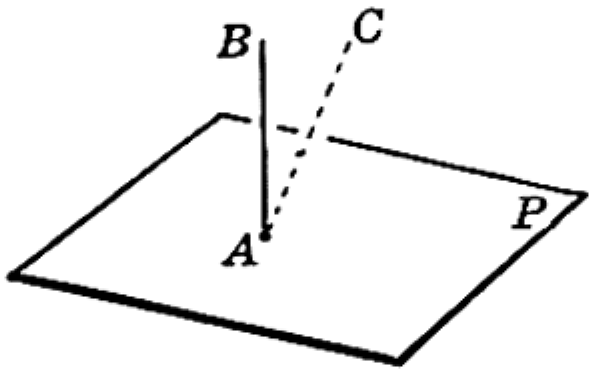
1. Draw c, any line in P.
2. Through K draw plane $Q \perp c$, intersecting P in e. (§ 31)
3. In Q draw line a through $K \perp e$.
4. $a \perp P$.

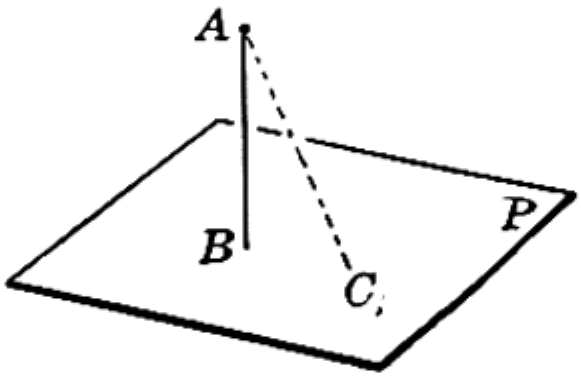
PROOF:

Draw lines b and d in planes P and Q respectively, completing the skew quadrilateral $abcd$.

Statements	*Reasons*
1. $c \perp Q$.	1. Cons. 2.
2. $c \perp d$, $c \perp e$.	2. Def. ⊥ to plane.
3. $a \perp e$.	3. Cons. 3.
4. $a \perp b$.	4. Skew quad. theorem. (§ 25)
5. $a \perp P$.	5. A line ⊥ two intersecting lines at pt. of intersection is ⊥ plane of lines.

37. THEOREM. Through a given point only one perpendicular can be drawn to a given plane.





GIVEN: Pt. A in or outside plane P.
$AB \perp P$.

PROVE: No other line through A is $\perp P$.

Following the indirect method in § 33, assume that another perpendicular AC is possible. Pass a plane through AB and AC and show that a proposition in plane geometry is contradicted.

Write out the proof.

EXERCISES

1. a. On the assumption that the faces of a rectangular solid are rectangles (§ 6), prove that each edge of the solid is perpendicular to the face that it intersects.
  b. Prove that the edge of a room in which two walls intersect is perpendicular to one of the diagonals of the floor.

2. a. Prove that the diagonals of a rectangular solid are equal. What is the analogous proposition in plane geometry?
  b. One edge of a cube is 5 feet. Find the length of the diagonal of the cube in terms of the simplest radical.

3. a. Prove that there is only one line perpendicular to each of two intersecting lines.
  b. Prove that each of three mutually perpendicular lines is perpendicular to the plane determined by the other two.

4. a. Prove: If a right angle is rotated about one side as an axis, the other side generates a plane perpendicular to the axis.
  b. Using his square, how can a carpenter test whether an upright post is perpendicular to the floor?

5. Line AB is perpendicular to plane P at B. Points C and D are in P.
  a. If $BC = 12$ and $BD = 12$, find AC and AD in terms of AB.
  b. If $AC = 15$ and $AD = 15$, find BC and BD in terms of AB.

6. One side of a square $ABCD$ is $8\sqrt{2}$ inches. The diagonals of the square intersect at O. OE is perpendicular to the plane of the square and is 6 inches long. Find the area of $\triangle AEB$ in terms of the simplest radical.

7. Through the point where two lines a and b intersect, two planes P and Q are drawn perpendicular to a and b respectively. P and Q intersect in c. Prove that c is perpendicular to the plane of a and b.

8. Using the skew quadrilateral theorem (§ 25), prove the following proposition about four perpendiculars:

   If from the foot of a perpendicular to a plane a second line is drawn perpendicular to a third line in the plane, then any fourth line from the point of intersection of the second and third lines to a point in the perpendicular (the first line) to the plane is also perpendicular to the third line.

9. A line AB is tangent to a circle O at C. At O, the center of the circle, OD is drawn perpendicular to the plane of the circle. If E is any point on OD, prove that the line EC is perpendicular to AB.

10. If a line is perpendicular to a plane, prove that all the perpendiculars from the line to a line in the plane are concurrent.

11. **a.** A pole 12 feet high is steadied by three guy wires from its top to stakes on the ground. One of these stakes is 9 feet from the pole and the wire to it is 15 feet long. Another stake is 16 feet from the pole and the wire to it is 20 feet long. Can you tell from this if the pole is upright? Why?

    b. Suppose that the pole in (a) is held up by four guy wires. Could you tell if the pole is upright from the given measurements? Why?

    c. Suppose that the 12-foot pole in (a) is held up by four guy wires. The only other measurements you know are that one stake is 11 feet from the pole and the wire to it is 16 feet long. Can you tell from this if the pole is upright? Why?

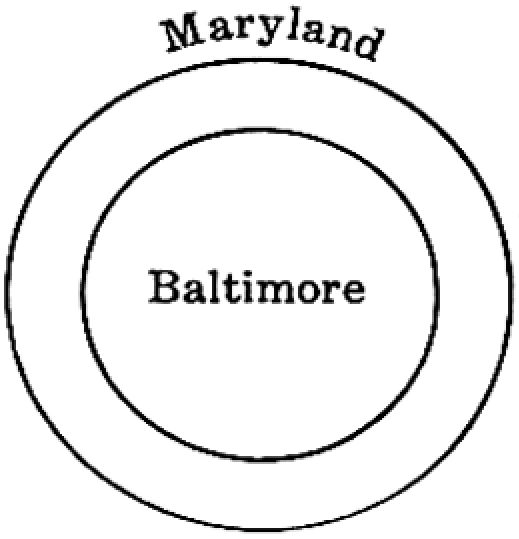
SUPERIOR WORK

S 1. We saw that the first statement in an indirect proof is that one of several propositions is true. In its simplest form, it states that either A or B is true. This is called an *alternative proposition* or a *disjunction*.

Let us consider again the proposition A, its opposite-converse C, and their illustrative diagram from page 20.

A: If a building is in Baltimore, it is in Maryland.

C: If a building is outside Maryland, it is outside Baltimore.



The following alternative proposition is evident.

D: A building is either in Maryland or outside Maryland.

Using C, proposition D becomes

E: A building is either in Maryland or outside Baltimore (or both).

Every conditional proposition A has an equivalent alternative proposition E. Thus the assumption that if a thing is true it can be proved leads to the alternative proposition that a thing can either be proved or it is false.

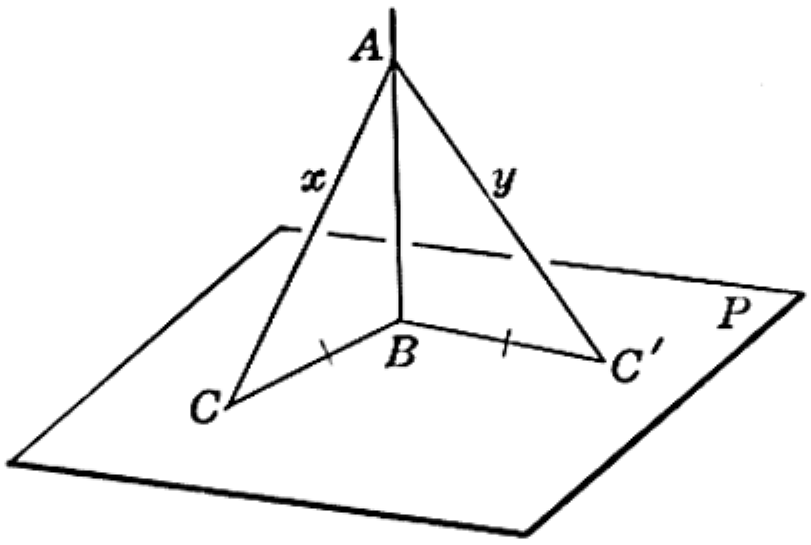
State the alternative propositions that are equivalent to the propositions in Exercises 4–9 on pages 2–3.

S 2. Using the skew quadrilateral theorem (§ 25), prove that, if a line is perpendicular to each of two oblique lines at their point of intersection, it is perpendicular to the plane of these lines.

LINE SEGMENTS FROM A POINT TO A PLANE

How many equal line segments can be drawn from a point to a line? from a point to a plane? The following theorem concerns these line segments.

38. THEOREM. If from a point in a perpendicular to a plane line segments meet the plane at equal distances from the foot of the perpendicular, they are equal; conversely, if the line segments to the plane are equal, they meet the plane at equal distances from the foot of the perpendicular.



GIVEN: $AB \perp$ plane P.
A is any pt. on AB.
C and C' are in P.
$BC = BC'$.

PROVE: $AC = AC'$.

Write out the proof of this and also the proof of the converse.

39. (Optional) **THEOREM. If from a point in a perpendicular to a plane, line segments meet the plane at unequal distances from the foot of the perpendicular, they are unequal in the same order; conversely, if the line segments to the plane are unequal, they meet the plane at distances from the foot of the perpendicular that are unequal in the same order.**

In the figure above:

1. $AB^2 = AC^2 - BC^2$ and $AB^2 = AC'^2 - BC'^2$. (Pythagorean Theorem)
2. Hence $AC^2 - BC^2 = AC'^2 - BC'^2$. (Subs. Ax.)

3. If $BC > BC'$. $BC^2 > BC'^2$,
4. and $AC^2 > AC'^2$, (§ 442)
or $AC > AC'$.

In the same way prove the converse.

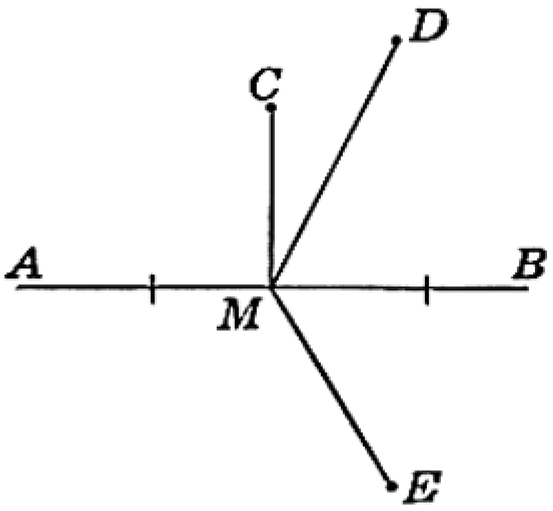
40. (Optional) **Corollary. Of all the line segments that can be drawn from an external point to a plane the perpendicular is the shortest.**

41. Definition. *The distance from a point to a plane is the length of the perpendicular from the point to the plane.*

A Point Equidistant from Two Given Points

42. In plane geometry the two-point theorem states that two points each equidistant from the ends of a line segment determine the perpendicular bisector of the line segment. Analogously, in solid geometry we have the three-point theorem.

43. THREE-POINT THEOREM. Three non-collinear points each equidistant from the ends of a line segment determine the plane that bisects the line segment and is perpendicular to it.



GIVEN: Non-collinear pts. C, D, and E each equidistant from A and B.

PROVE: C, D, and E determine the plane that bisects AB and is $\perp$ to AB.

CONS.: Join M, the midpoint of AB, to C, D, and E.

PROOF: *Statements*	*Reasons*
1. AB and MC determine a plane.	1. Two intersecting lines determine a plane.
2. MC $\perp$ bisector of AB. Similarly, MD and ME are $\perp$ bisectors of AB.	2. Two-pt. Theorem. (§ 449)
3. MC and MD determine the plane P that is the $\perp$ bisector of AB.	3. Two lines $\perp$ same line at same pt. determine the plane $\perp$ the line at that pt. (§ 34)
4. ME is in P.	4. All the $\perp$s to a given line at a given pt. lie in a plane $\perp$ the given line at that pt. (§ 35)

44. Corollary. If a point is equidistant from the ends of a line segment, it lies in the plane that is perpendicular to the line segment and bisects it (since this point and two more like it determine the plane).

45. CONVERSE THEOREM. Every point in the plane that is the perpendicular bisector of a line segment is equidistant from the ends of the line segment.

Draw a figure and write out the proof of this theorem.

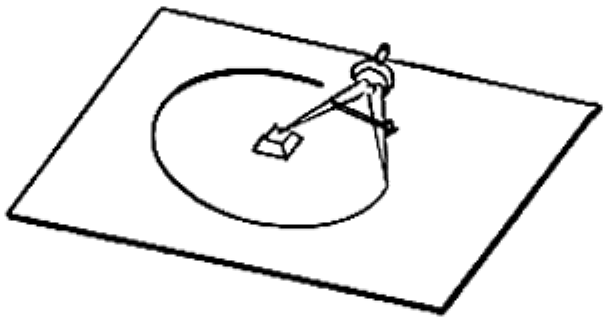
The two preceding propositions may be summarized by the following locus theorem. (§ 365)

46. LOCUS THEOREM. The locus of a point in space that is equidistant from the ends of a line segment is the plane that is the perpendicular bisector of the line segment.

What is the analogous theorem in plane geometry?

EXERCISES

1. **a.** What is the locus of a point in a plane that is equidistant from a point in another plane? Why?
   b. Why is it possible to draw a circle in a plane by setting one leg of the compass at a point outside the plane?



2. Prove:
   a. If two equal line segments are drawn from a point to a plane, they make equal angles with the perpendicular from the point to the plane.
   b. If two lines from a point to a plane make equal angles with the perpendicular from that point to the plane, they meet the plane at equal distances from the foot of the perpendicular.
3. At the midpoint of the hypotenuse of a right triangle a perpendicular is drawn to the plane of the triangle. Prove that any point on this perpendicular is equidistant from the vertices of the triangle.
4. A line segment and its perpendicular bisector lie in a certain plane. Through a point on the perpendicular bisector a perpendicular to that plane is drawn. Prove that any point on the perpendicular to the plane is equidistant from the ends of the line segment.
5. Prove that any point on the perpendicular to the plane of a circle through its center is equidistant from two equal chords of the circle.

6. Through the midpoint of one base of a trapezoid a perpendicular is drawn to the plane of the trapezoid. Equal line segments join a point on the perpendicular with the ends of the other base. Prove that the trapezoid is isosceles.

7. The vertices of a square are equidistant from a point outside their plane. Prove that a line through this point and the center of the square is perpendicular to the plane of the square.

8. **a.** Is the two-point theorem (§ 449) true in space?
   b. What is the locus of the vertices of the isosceles triangles in space that have a common base?
   c. Describe the locus of points on the floor of a room that are equidistant from two points in each of two walls.

9. **a.** Locate a point in a plane that is a given distance from a point in the plane and also equidistant from two points outside the plane.
   b. Locate a point in a plane that is equidistant from two points in the plane and also equidistant from two points outside the plane.
   c. Locate points in a plane that are equidistant from two intersecting lines in the plane and also equidistant from two points outside the plane.

10. Equal line segments join three given points in a plane to a point outside the plane. Prove that the foot of the perpendicular to the plane through the outside point is the center of the circle through the three given points.

SUPERIOR WORK

S 1. Prove: If a line makes equal angles with three coplanar lines at their point of intersection, it is perpendicular to the plane of the lines. *Suggestion:* Use Exercise 10.

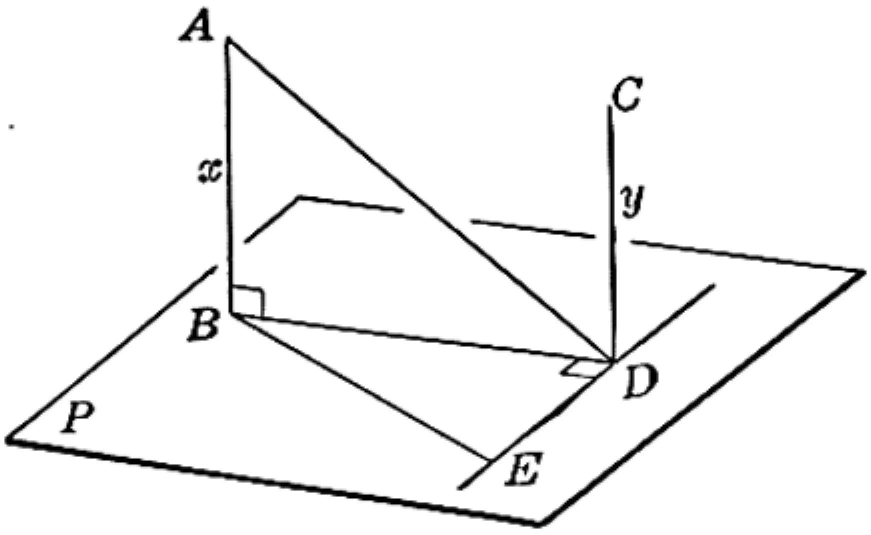
S 2. A point O is not in the same plane with rectangle $ABCD$. Prove: $OA^2 + OC^2 = OB^2 + OD^2$.

TWO PERPENDICULARS TO A PLANE

47. Let us move our perpendicular to a plane and consider two positions. By joining the feet of the two perpendiculars we shall have two lines x and y perpendicular to the same line. In plane geometry x and y would be parallel. To prove that x and y are also parallel in space, we must first show that they are in the same plane.

The proof shows that the three lines BD, AD, and CD are perpendicular to the same line ED at the same point D. From Corollary § 35 we see that these three lines are in one plane. Note that BE is drawn to complete the skew quadrilateral.

48. THEOREM. Two perpendiculars to the same plane are parallel.



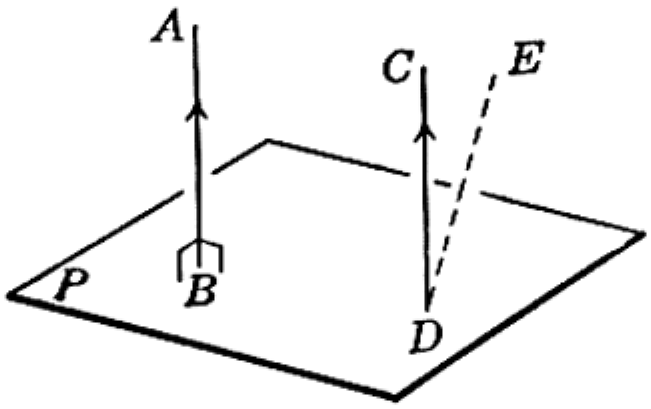
GIVEN: $AB \perp$ plane P at B.
$CD \perp P$ at D.

PROVE: $AB \parallel CD$.
CONS.: Draw BD.

In P draw $DE \perp BD$.
Draw BE and AD.

PROOF: *Statements*	*Reasons*
1. $AB \perp BD$, $AB \perp BE$.	1. Def. $\perp$ to plane.
2. $BD \perp DE$.	2. Cons.
3. In skew quad. $ABED$, $AD \perp DE$.	3. Skew quad. theorem. (§ 25)
4. $CD \perp DE$.	4. Def. $\perp$ to plane.
5. BD, AD, and CD are in one plane.	5. All $\perp$s to a given line at a given pt. are in a plane $\perp$ to given line at given pt. (§ 35)
6. AB lies in same plane as CD.	6. Def. plane.
7. $CD \perp BD$.	7. Def. $\perp$ to plane.
8. $AB \parallel CD$.	8. Two lines in same plane $\perp$ same line are $\parallel$. (§ 459)

49. CONVERSE THEOREM. If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to that plane.



GIVEN: $AB \parallel CD$.
$AB \perp$ plane P.

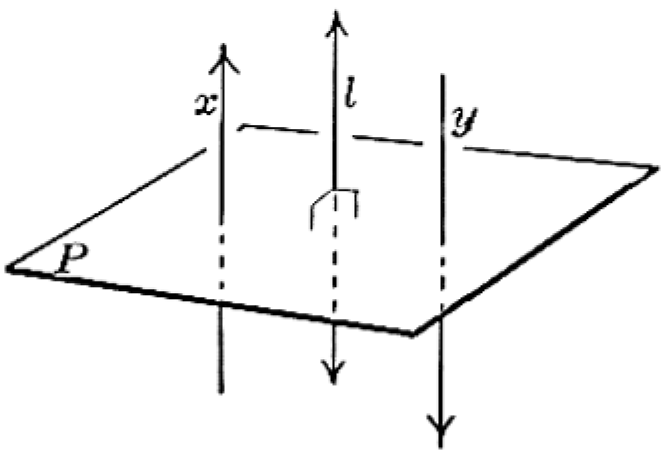
PROVE: $CD \perp P$.

INDIRECT PROOF: *Statements*	*Reasons*
1. Either $CD \perp P$ or not.	
2. Assume CD not $\perp P$.	
3. Then draw $DE \perp P$.	3. Through a given pt. a $\perp$ can be drawn to a given plane. (§ 36)

4. $DE \parallel AB$.	4. $\perp$s to same plane are $\parallel$. (§ 48)
5. This is impossible.	5. Through a given pt. only one $\parallel$ can be drawn to a given line. (§ 540)
6. Hence, $CD \perp P$.	

In plane geometry, if two lines are parallel to a third line, they are parallel to each other. This is true also in space, but the proof must be more general than the one in plane geometry.

50. THEOREM. If two lines are parallel to a third line, they are parallel to each other.



GIVEN: $x \parallel l$. $y \parallel l$.

PROVE: $x \parallel y$.

CONS.: Draw plane $P \perp l$.

Write out the proof.

51. Corollary. A line parallel to one of two skew lines cannot be parallel to the other.

EXERCISES

1. Prove that the opposite edges of a rectangular solid are parallel.
2. Prove that any two diagonals of a rectangular solid bisect each other.
3. Prove that a diagonal of a cube cannot be perpendicular to a plane which contains another diagonal of the cube.
4. Prove:
   a. If a plane bisects a line segment, the ends of the line segment are equidistant from the plane.
   b. If a plane passes through one diagonal of a parallelogram, it is equidistant from the ends of the other diagonal.
5. Prove: If a plane bisects two sides of a triangle, then any point on the third side is the same distance from the plane as the opposite vertex of the triangle.
6. Prove: If two points are equidistant from a plane and on opposite sides of it, then the plane bisects the line segment connecting the two points.
7. Prove: If the midpoints of the sides of a skew quadrilateral are joined in succession, a parallelogram is formed.

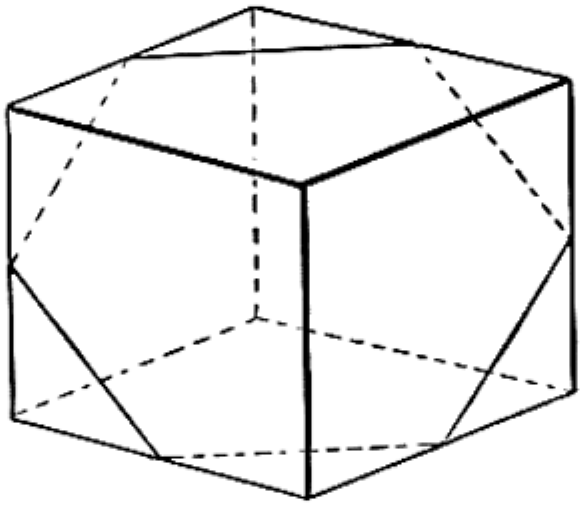
8. Prove that the line segments that connect the midpoints of the opposite sides of a skew quadrilateral bisect each other.

9. Prove that all the perpendiculars to a plane through a given line are coplanar (§ 21).

10. Prove: If a line segment does not intersect a plane, the distance from the midpoint of the line segment to the plane is the average of the distances from the ends of the line segment to the plane.

SUPERIOR WORK

S 1. In the figure the midpoints of six successive edges of a cube are joined by line segments that are in successive planes. What kind of polygon do the line segments form? Is the polygon coplanar? Copy the figure and prove your answers.



S 2. Prove that four points are not necessarily coplanar. *Suggestion:* Assume the opposite and show that all space would then be one plane.

SUMMARY

52. In solid geometry we study figures that involve more than one plane. To represent such figures on paper we use perspective. The propositions about these figures are based on propositions in plane geometry. We therefore reviewed some basic plane geometry by representing the figures in perspective.

Some propositions in plane geometry have to do with figures that can always be contained in one plane. These propositions may be used where they apply in solid geometry without further proof. Other propositions concern figures that may occupy more than one plane. Some of these are not true in space, while the others require further proof before they may be used.

Constructions in solid geometry are based on those in plane geometry, but they are not actually performed on paper with compass and straightedge because the figures are not in their true shape. In solid geometry, constructions are in the form of theorems proving what can be done by making certain constructions in certain planes.

A plane is defined in terms of straight lines and points. It is determined by three non-collinear points, by a straight line and an external point, by two intersecting lines, or by two parallel lines. To help us emerge from the plane, we used a quadrilateral composed of two right triangles with a common leg. By folding this figure along the common leg we found four pairs of perpendicular lines in space. This led us to the definition of a perpendicular to a plane and helped us to prove some propositions about it.

If a line is perpendicular to a plane, it is necessarily perpendicular to every line in the plane passing through its foot; but for a line to be perpendicular to a plane it is sufficient that it be perpendicular to only two lines in the plane through its foot. Through a given point, a plane perpendicular to a given line, or a line perpendicular to a given plane, can be constructed. These constructions are unique.

We extended to space the two-point theorem and the theorem about the locus of a point equidistant from two given points.

If the distances from a point on a perpendicular to a plane to two points in the plane are equal, the distances from the points in the plane to the foot of the perpendicular are equal; and conversely.

Perpendiculars to the same plane are parallel. A converse of this enabled us to use a perpendicular plane as a "transversal" and extend the following theorem from plane geometry to solid geometry: If two lines are parallel to a third line, they are parallel to each other.

RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, go back to the section indicated.

solid geometry § 1
intuitive fact § 2
surface § 3, § 7
perspective § 3
  convergence
  vanishing point
  picture plane
rectangular solid § 6
  cube

plane § 8
intersection of figures § 14
foot of line § 16
skew lines § 20
skew quadrilateral § 21
coplanar § 21
perpendicular to plane § 27
plane perpendicular to line § 28
distance from point to plane § 41

REVIEW EXERCISES

1. Which of the following figures are necessarily coplanar? A triangle, a quadrilateral, a trapezoid, a parallelogram, a circle, an arc of a circle, a circle and its tangents, three parallels and a transversal of them, three concurrent lines, a circle and a perpendicular to a diameter at one end.

2. **a.** If three planes pass through a common point, do their lines of intersection pass through that point? Why?

b. If an oblique angle is rotated about one side as an axis, does the other side generate a plane?

c. If a line is oblique to a plane, is it oblique to every line in the plane through its foot?

3. **a.** Show how to make four equilateral triangles using six match sticks.
  b. Three books, each one inch thick and each containing 1000 pages, are standing on a shelf. Volume I is on the left and Volume III is on the right. A bookworm bores from Vol. I, page 1, to Vol. III, page 1000. How many inches long is the bore?

4. Describe the possible cross section made by a plane that intersects each of the following figures: (*a*) a line perpendicular to a plane; (*b*) two intersecting planes; (*c*) two parallel lines intersecting a plane.

5. **a.** What part of the definition of adjacent angles (§ 376) would indicate that they must be in one plane?
  b. What part of the definition of an external or an internal tangent (§ 421) would imply that the figure is in one plane?
  c. Modify the definition of tangent circles (§ 422) so that their line of centers would always pass through the point of contact.

6. Assuming that a line and an external point determine a plane, prove that three non-collinear points determine a plane.

7. On the assumption that intersecting planes must have more than one point in common, prove that they intersect in a straight line.

8. Draw a square in perspective. Show how to locate the center of an inscribed circle and the points of tangency.

9. Draw a rectangle in perspective. Show how to divide one of its sides into two equal parts; into four equal parts.

10. Using the indirect method, prove that four lines cannot be mutually perpendicular.

11. Prove: If a line is perpendicular to a diameter at one end and is parallel to a chord, it is tangent to the circle.

12. Prove that a plane through the center of a circle perpendicular to a chord bisects the chord and its arcs.

13. Prove that a plane perpendicular to a tangent at the point of contact passes through the center of the circle.

14. In space, concentric circles are defined as circles that have the same center and are in one plane. Prove that, if two skew circles with the same center are cut by a plane perpendicular to the intersection of the planes of the circles, the four points of intersection can be made the vertices of a parallelogram.

15. On the three edges of a rectangular solid that meet in O, points A, B, and C are located so that $OA = 15$ inches, $OB = 20$ inches, and $OC = 20$ inches. Find the area of $\triangle ABC$.

16. From a common point O, line segments OA, OB, and OC are each 10 inches long. $\angle AOB = 60°$, $\angle BOC = 60°$, and $\angle COA = 90°$. Find the area of $\triangle ABC$.

17. Prove: If one side of a triangle is in a plane, the midpoints of the other two sides are equidistant from the plane.

18. The vertices of $\triangle ABC$ are joined to a point O outside the plane. $OA = 6$, $OB = 9$, $OC = 13\frac{1}{2}$, $AB = 4$, and $BC = 6$. Prove $\angle AOB = \angle BOC$.

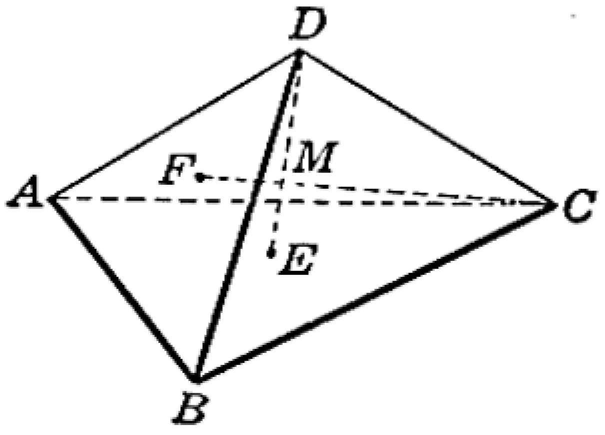
19. Draw a circle with six radii. Cut out the circle and cut away one of the central angles. Fold the figure along the radii until the two straight cuts fit together. What does your figure show about the sum of the angles about a point in space?

20. Draw two lines making an angle of 60°. Fold your paper along one of these lines. Will the segments of the other line ever become perpendicular?

Repeat the experiment, making the angle between the lines 30°, 44°, and 46°.

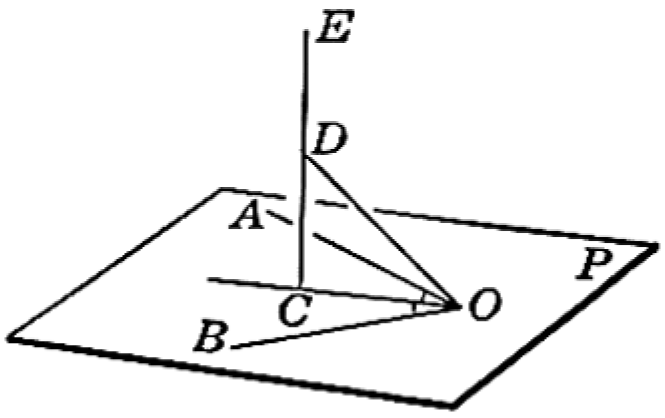
21. In the figure the vertices of $\triangle ABC$ are joined to D, a point outside the plane. (The figure is called a tetrahedron.) E is the centroid (the intersection of the medians) of $\triangle ABC$; F is the centroid of $\triangle ABD$. Prove:

a. DE and CF meet in some point M.
b. $EF = \frac{1}{3}CD$. (§ 482)
c. $EM = \frac{1}{4}ED$



22. The vertices of a $\triangle ABC$ are joined to O, a point outside the plane. A', B', and C' are the midpoints of OA, OB, and OC respectively. Prove: $\triangle A'B'C' \sim \triangle ABC$.

23. In the figure, $\angle AOB$ is in plane P, and C is any point in the bisector of $\angle AOB$. $CE \perp P$. Prove: A line from O to any point D in CE makes $\angle DOA = \angle DOB$.



24. In the figure for Exercise 23 draw $CF \perp OA$ at F. Prove that a line from F to any point D in CE is perpendicular to OA.

25. In the figure for Exercise 23 prove that any point D in CE is equidistant from the sides of $\angle AOB$.

SUPERIOR WORK

S 1. In an alternative proposition both statements cannot be false. In other words, the falsity of both statements is impossible or *incompatible*. An alternative proposition may therefore be stated as an incompatible relationship. Thus, the alternative proposition that either A or B is true may be stated that the falsity of A is incompatible with the falsity of B. We see that the opposite of an alternative proposition is an incompatible relationship.

On page 28, the proposition that if a building is in Baltimore it is in Maryland led us to the alternative proposition: A building is either in Maryland or not in Baltimore. This, in turn, may be stated as: Not being in Maryland is incompatible with being in Baltimore.

Write the following propositions as alternative propositions and as incompatible relationships:

a. If two lines are skew, they are not parallel.
b. If two lines intersect, they are not skew.
c. If a thing can be proved, it is true.
d. To be free, a people must have democracy.

S 2. a. Show that an irresistible force and an immovable object are incompatible.
b. Demosthenes wrote that all Greeks are liars. Show that this is an incompatible statement, considering that Demosthenes was a Greek.

S 3. In logic the symbol a/b denotes that a is incompatible with b. Translate into simple English the following symbols:

a. a/a **b.** $a/(b/b)$ **c.** $(a/a)/(b/b)$

TEST

1. Prove: If a plane cuts a line segment, the two parts are proportional to the distances from the ends of the line segment to the plane.

2. A diagonal of a cube is 12 inches. Find the area of the quadrilateral formed by joining the ends of two diagonals of the cube.

3. Prove: A plane perpendicular to a chord of a circle at its midpoint bisects all the chords parallel to the perpendicular chord.

4. Prove that a line cannot be perpendicular to two intersecting planes.

5. Describe how to construct three mutually perpendicular lines.

6. Given non-collinear points O, A, and B. Tell how to construct a line OC such that the perpendiculars from A and B to OC will meet. Is the construction unique?

7. a. What fact in solid geometry does a mason illustrate when he levels the top of a concrete walk?
b. What proposition is illustrated by a swinging door?
c. What fact is illustrated by the spokes of a wheel and its axle?
d. Are two meshed gears always coplanar?
e. When are the wheels of a bicycle coplanar? When are they skew?
f. What is the locus of a point equidistant from the vertices of a triangle?
g. What is the locus of a point on the floor of a room that is a given distance from a point on the wall?

8. Which of the following statements are true and which are false?

   a. Three points determine a plane.
   b. If a line cuts three concurrent lines, all four lines are in one plane.
   c. If the vertices of a quadrilateral are skew, the midpoints of the sides of the quadrilateral are skew.
   d. All collinear points are coplanar and conversely.
   e. If two skew lines are cut by two transversals, the transversals are not necessarily skew.
   f. The intersection of two surfaces is a straight line.
   g. In a perspective drawing, parallel lines are not necessarily convergent.
   h. In a perspective drawing, equal lines are sometimes represented by equal lines.
   i. If a line is perpendicular to a line in a plane through its foot, it is perpendicular to the plane.
   j. If a line is perpendicular to a plane, it is perpendicular to every line in the plane.
   k. If a line is perpendicular to two lines, it is perpendicular to the plane of the lines.
   l. If a line is oblique to one line in the plane through its foot, it is oblique to the plane.
   m. If two lines are oblique to a plane, they are not parallel.
   n. If one of two lines is oblique to a plane and the other is perpendicular to the plane, the lines are not parallel.
   o. Two line segments joining two points in a plane with a point in a perpendicular to the plane are proportional to the distances of the two points from the foot of the perpendicular.

9. Tell whether each of the following statements is true in plane geometry, in solid geometry, or in both.

   a. If two lines are not parallel, a perpendicular to one of them is not perpendicular to the other.
   b. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.
   c. All the points equidistant from the sides of an angle are not necessarily on the bisector of the angle.
   d. The angle bisectors of a triangle meet in a point.
   e. The perpendicular bisectors of the sides of a triangle meet in a point.
   f. An equilateral quadrilateral is not always a rhombus.
   g. If the opposite sides of a quadrilateral are equal, the diagonals may not bisect each other.
   h. If the opposite sides of a quadrilateral are equal and if the diagonals are equal, the figure is a rectangle.
   i. It is possible for a line to turn continuously and return to its original position without having rotated through 360°.
   j. If the vertex of an angle is at the center of a circle, the angle is a central angle of the circle.

k. If the hypotenuse of a right triangle is a diameter of a circle, the triangle is inscribed in the circle.

l. The perpendicular bisector of a chord bisects its arc.

m. If a line is perpendicular to a diameter at one end, it does not cut the circle.

n. If a line that is parallel to a tangent of a circle cuts the circle at one point, then it cuts the circle at another point.

o. If a figure is made up of two right triangles with a common leg, it is a quadrilateral.

10. Complete the following statements.

a. A straight line and two points determine at most —?— planes.

b. If two planes have three —?— points in common, they coincide.

c. If a line and a plane have —?— points in common, the line is in the plane.

d. Three planes can have a(n) —?— or a(n) —?— in common.

e. If the vertices of a quadrilateral are skew, the diagonals are —?—.

f. In perspective a circle is usually drawn as a(n) —?—.

g. If a line moves so that it remains parallel to its original position and always intersects a given line, it generates a(n) —?—.

h. The ends of two skew line segments determine —?— planes.

i. To determine that a board is horizontal, a carpenter must use his level at least —?— time(s).

j. If a line is oblique to a plane, it is perpendicular to —?— line(s) in the plane.

k. If a plane is oblique to one of two —?— lines, it is oblique to the other.

l. If a plane quadrilateral is folded along one diagonal, the space diagonal of the skew quadrilateral is —?— the folded diagonal.

m. If line a is coplanar with line b, and b is coplanar with line c, then a is not —?— coplanar with c.

n. One of the following three statements about lines a and b is true and the other two are false: a and b intersect; a and b are —?—; a and b are —?—.

o. The three planes that bisect the sides of a triangle and are perpendicular to them meet in a —?—.

2 Assembling Planes

Parallel Planes and Lines

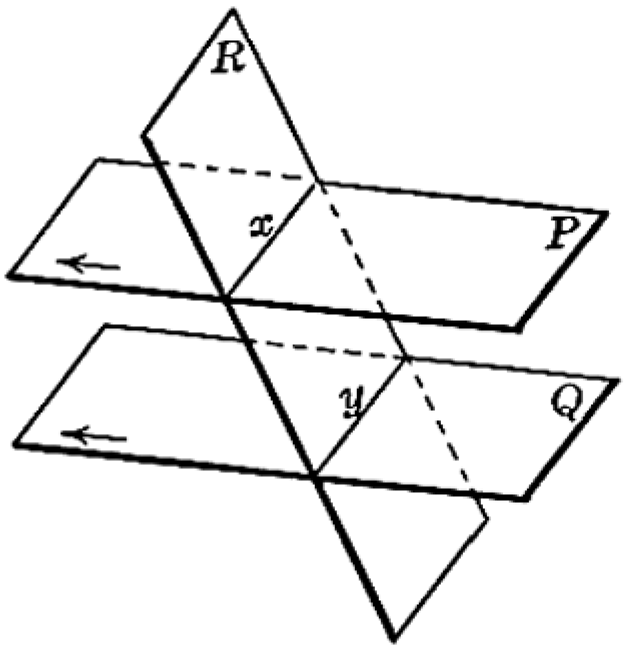
PARALLEL PLANES

In this chapter we shall study figures involving more than one plane. In the first lesson we shall be concerned with two parallel planes and their transversals.

53. Definition. *Parallel planes are planes that do not meet however far they are extended.*

If the transversal of two parallel planes is another plane, we have the following theorem.

54. THEOREM. If two parallel planes are cut by a third plane, the intersections are parallel.



GIVEN: Plane $P \parallel$ plane Q.
Plane R intersects P and Q in x and y respectively.

PROVE: $x \parallel y$.

Show that if x were not parallel to y, then P and Q would meet.

Write out the indirect proof.

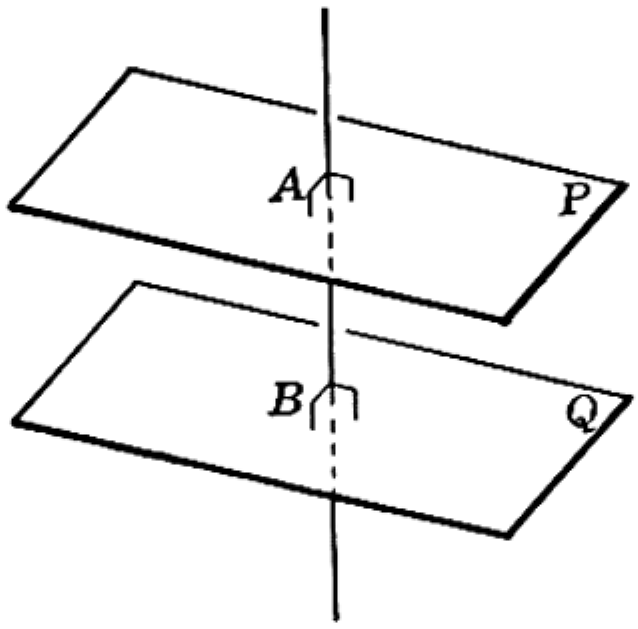
55. Corollary 1. Parallel line segments included between parallel planes are equal.

Use the plane of the parallel line segments as a transversal.

56. Corollary 2. Parallel planes are everywhere equally distant.

We next consider a perpendicular line as a transversal to two planes.

57. THEOREM. If two planes are perpendicular to the same line, they are parallel.



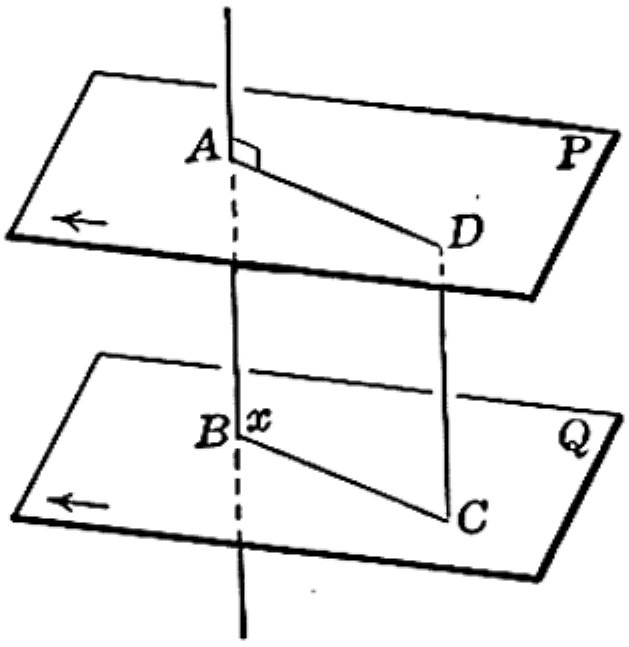
GIVEN: Plane $P \perp AB$.
Plane $Q \perp AB$.

PROVE: $P \parallel Q$.

If P and Q met, it would contradict § 33.

Write out the indirect proof.

58. CONVERSE THEOREM. If a line is perpendicular to one of two parallel planes, it is perpendicular to the other.



GIVEN: Plane $P \parallel$ plane Q.
$AB \perp P$ at A.
AB intersects Q at B.

PROVE: $AB \perp Q$.

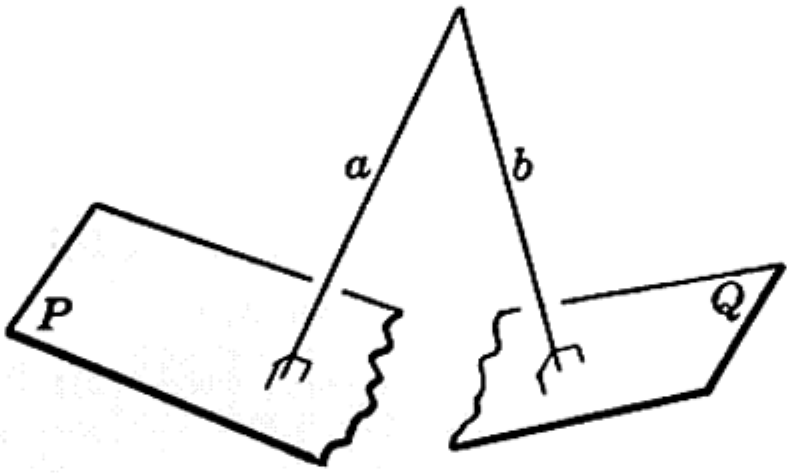
CONS.: Through B draw *any* line BC in Q.
Pass a plane through AB and BC intersecting P in AD.

PROOF:

1. $AB \perp P$.
2. $AB \perp AD$.
3. $AD \parallel BC$.
4. $AB \perp BC$.
5. $AB \perp Q$.

Supply the reason for each step.

59. Corollary. If two planes are perpendicular to two intersecting lines, they cannot be parallel.



If P were parallel to Q, then b would be perpendicular to P (Why?) and a would be parallel to b (Why?).

Theorem § 57 affords a method of constructing a plane parallel to a given plane.

60. Theorem. Through a given external point one plane, and only one, can be passed parallel to a given plane.

a. Through the given point construct a line perpendicular to the given plane (§ 36). Then construct a plane perpendicular to the line at the given point (§ 31).

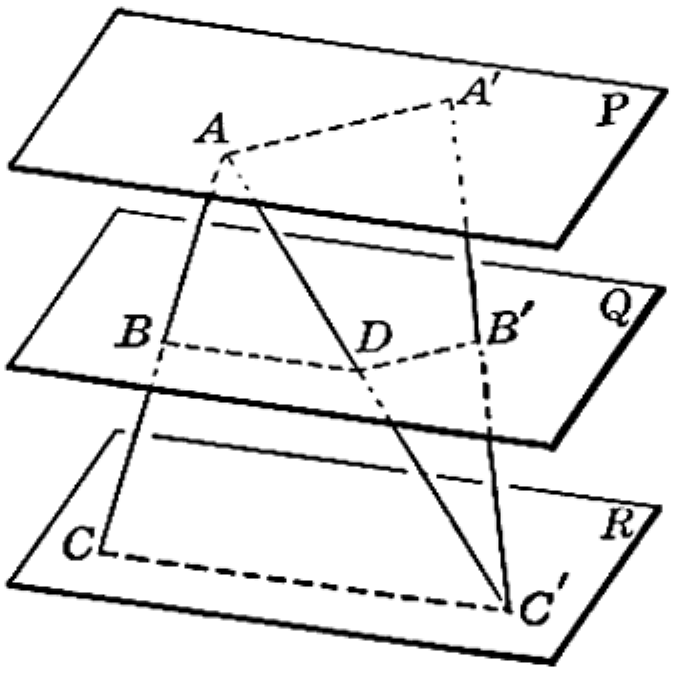
Draw a figure and describe the construction.

b. Assume that more than one such plane can be drawn. Through the given point draw a perpendicular to the given plane and show that it would contradict § 33.

Draw a figure and write out the indirect proof.

Analogous to the theorem in plane geometry about three parallels cut by two transversals is the following theorem in space.

61. THEOREM. If three parallel planes are cut by two lines, the segments on these lines are proportional.



GIVEN: Parallel planes P, Q, and R cut by transversals AC and $A'C'$.

PROVE: $\frac{AB}{BC} = \frac{A'B'}{B'C'}$.

CONS.: Draw AC'. Pass a plane through AC and AC' cutting Q in BD and R in CC'.

Pass a plane through $A'C'$ and $C'A$ cutting P in AA' and Q in DB'.

Apply § 465 to $\triangle ACC'$ and $A'C'A$.

Write out the proof.

62. Corollary. If three parallel planes cut equal segments on one transversal, they cut equal segments on any transversal.

EXERCISES

1. Prove: If two parallel planes cut the sides of an angle, the lines joining in succession the four points of intersection form a trapezoid.
2. If two intersecting line segments are included between two parallel planes, they divide each other proportionally.

3. Assuming that a cube is a solid enclosed by six squares (§ 6), prove that the opposite faces of a cube are parallel.

4. Prove: If two planes are each parallel to a third plane, they are parallel to each other. *Hint:* Draw a perpendicular to one plane.

5. If two planes are parallel, prove that any third plane that intersects one of them will intersect the other.

6. If a rectangle is rotated about one side as an axis, prove that the two sides adjacent to the axis will generate parallel planes.

7. What is the locus of a point 4 feet from a given plane?

8. Prove that no line can be perpendicular to each of two intersecting planes. Use § 57.

9. Prove that if two lines are perpendicular respectively to two intersecting planes, the lines are not parallel.

10. The following refer to the figure in § 61:
    a. $AB = 6$, $BC = 8$, $A'B' = 7$. Find $B'C'$.
    b. $AB = 5$, $A'B' = 6$, $B'C' = 9$. Find BC.
    c. $AB = m$, $BC = p$, $A'B' = 10$. Find $B'C'$.
    d. $AB = h$, $BC = h + 1$, $A'B' = h - 1$. Find $B'C'$.
    e. $AB = 8$, $BC = 10$, $A'C' = 20$. Find $B'C'$.
    f. $AB = \frac{3}{4}BC$, $A'C' = 21$. Find $A'B'$.

11. **a.** Find the locus of a point midway between two parallel planes.
    b. Find the locus of a point whose distances from two parallel planes are in the ratio 3:4.

12. After taking off from a carrier and ascending in a straight line a distance of $\frac{1}{4}$ mile, a seaplane is at an altitude of 300 feet. How much farther must the plane continue its climb to reach an altitude of 7500 feet?

SUPERIOR WORK

S 1. Two parallel planes are cut by three parallel lines. Prove that the triangle formed by joining the points of intersection in one plane is congruent to the triangle formed by joining the points of intersection in the other plane.

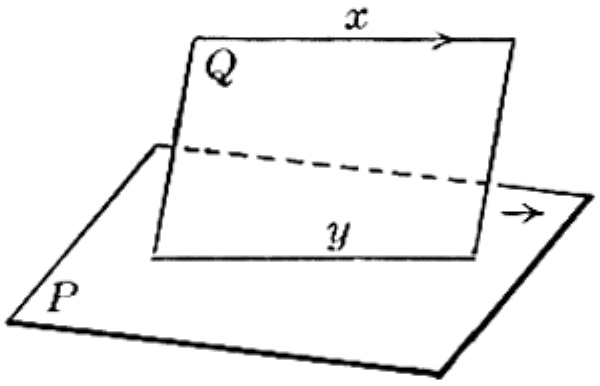
S 2. Two parallel planes are cut by three concurrent lines. Prove that the triangle formed by joining the points of intersection in one plane is similar to the triangle formed by joining the points of intersection in the other plane.

A LINE PARALLEL TO A PLANE

63. Definition. *A line and a plane are parallel if they do not meet no matter how far they are extended.*

If a line is perpendicular to a plane, a plane cross section containing the line consists of two perpendicular lines. Similarly, we shall prove that if a line is parallel to a plane, a planar cross section containing the line consists of two parallel lines.

64. THEOREM. If a line is parallel to a plane, it is parallel to the intersection of this plane with any plane containing the line.



GIVEN: Line $x \parallel$ plane P.
Plane Q contains x and cuts P in y.

PROVE: $x \parallel y$.

Show that if x and y intersect, then x and P must intersect.

Write out the indirect proof.

In the figure above, if P can rotate about a line l while remaining parallel to x, how is the line l related to x?

If x rotates while remaining parallel to P, does x remain in one plane? If so, how is this plane related to P?

The next two theorems will answer these questions.

65. THEOREM. If a line is parallel to two intersecting planes, it is parallel to their intersection.

Assuming that the proposition is false, pass a plane through the given line and some point on the intersection of the planes. Show that this would result in two intersecting lines being parallel to a third line.

Draw a figure and write out the indirect proof.

66. THEOREM. If a plane is parallel to each of two intersecting lines, it is parallel to the plane of these lines.

Assuming that the two planes meet, their intersection would be parallel to the two intersecting lines.

Draw a figure and write out the indirect proof.

EXERCISES

1. Prove: If a plane that is parallel to one side of a triangle cuts the other two sides, it divides them proportionally.
2. Prove: If two planes are parallel, any line in one plane is parallel to the other plane.

3. Prove: If two planes are parallel to a line, their intersections with any plane containing the line are parallel.

4. Prove: If a line is parallel to a plane, all the points on the line are equidistant from the plane.

5. Prove: All the lines through a given point that are parallel to a given plane are in a plane parallel to the given plane.

6. Prove: If a plane is parallel to a line, any other line parallel to that line through a point in the plane lies in the plane.

7. Prove: If two planes are parallel, a line through a point in one plane parallel to the other plane lies in the first plane.

8. Prove: If a line intersects one of two parallel planes, it will intersect the other.

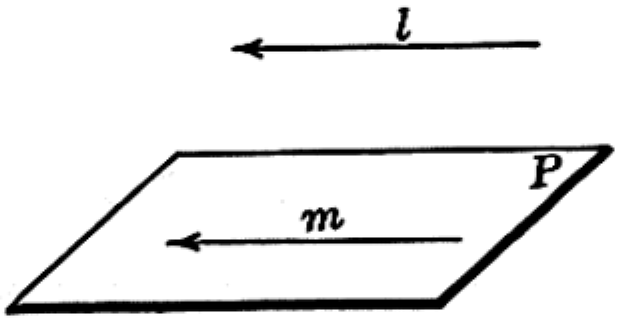
9. Prove: If two planes are parallel, a line parallel to one plane is parallel to the other. (Prove indirectly, using Exercise 8.)

10. Prove: If a line in one of two intersecting planes is parallel to the other plane, and a line in the second plane is parallel to the first plane, the two lines are parallel.

PARALLEL LINES

In this lesson we shall see that planes have certain relationships because they contain parallel lines.

67. THEOREM. If two lines are parallel, any plane which contains one line, and only one, is parallel to the other.



GIVEN: Line $l \parallel$ line m.
Plane P contains m but not l.

PROVE: $P \parallel l$.

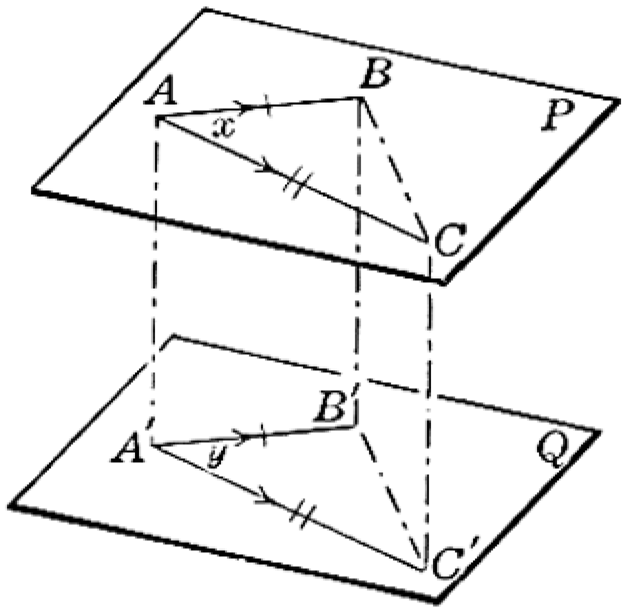
Show that if this were not true, the two parallel lines would meet.

Write out the indirect proof.

Can you point out how this is a converse of § 64?

If the two lines in the preceding theorem turn in such a way that they always remain parallel to each other, can they remain in parallel planes? Are the amounts of rotation at any moment the same? A consideration of these questions leads us to the following theorem.

68. THEOREM. If two angles not in the same plane have their sides respectively parallel and extending in the same directions, the angles are equal and their planes are parallel.



GIVEN: $AB \parallel A'B'$.
$AC \parallel A'C'$.
Plane P contains AB and AC.
Plane Q contains $A'B'$ and $A'C'$.

PROVE: $\angle BAC = \angle B'A'C'$.
$P \parallel Q$.

CONS.: Lay off $AB = A'B'$ and $AC = A'C'$.
Draw BC, $B'C'$, AA', BB', and CC'.

OUTLINE OF PROOF:

1. $ACC'A'$ and $ABB'A'$ are ▱s.
2. BB' and AA' are $\parallel$ and $=$.
   CC' and AA' are $\parallel$ and $=$.
3. BB' and CC' are $\parallel$ and $=$. (§ 50 and Subs. Ax.)
4. $BCC'B'$ is a ▱.
5. $BC = B'C'$.
6. $\triangle ABC \cong \triangle A'B'C'$.
7. $\angle BAC = \angle B'A'C'$.
8. $AB \parallel Q$, $AC \parallel Q$. (§ 67)
9. $P \parallel Q$. (§ 66)

Write out the complete proof.

Prove that if two angles have one pair of parallel sides in opposite directions and the other pair of parallel sides in the same direction, the angles are supplementary.

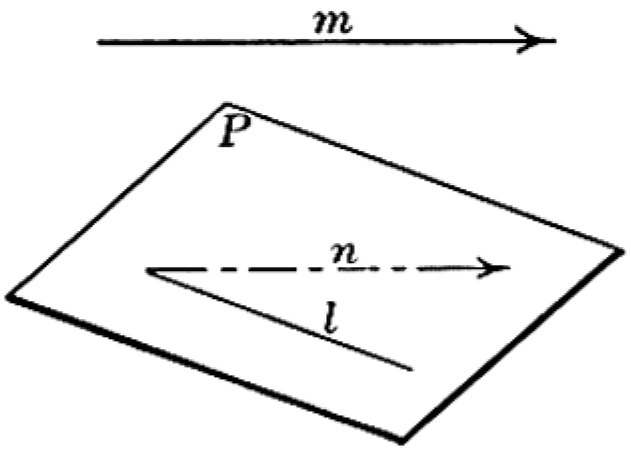
We now see that the following proposition from plane geometry is also true in space:

If two angles have their sides respectively parallel, they are either equal or supplementary.

In plane geometry we have a similar proposition about angles whose sides are respectively perpendicular. Is this true also in solid geometry?

The next two theorems will prove that it is possible to have a plane contain one of two skew lines and be parallel to the other; or contain a given point and be parallel to both skew lines.

69. THEOREM. Through one of two given skew lines, a plane, and only one plane, may be drawn parallel to the other.



GIVEN: Skew lines l and m.

PROVE: There is one plane passing through l and parallel to m.

1. Through any pt. on l, draw line n parallel to m.
2. Pass plane P through l and n.
3. $P \parallel m$. (Why?)

Prove that your construction is unique by using § 65 to show that if it were not, then l and m would be parallel.

70. THEOREM. Through any given point a plane, and only one plane, may be drawn parallel to each of two skew lines.

Suggestion: Through the given point draw two lines each parallel to one of the given skew lines. These will determine the required plane.

Draw a figure, write out the construction, and prove it.

Using the method of the preceding theorem, prove that your plane is unique.

EXERCISES

1. Prove: If a line is parallel to the intersection of two planes, it is parallel to each of the planes.
2. Prove: If two points on the same side of a plane are equidistant from the plane, the line through these points is parallel to the plane.
3. Prove: If a plane divides two sides of a triangle proportionally, it is parallel to the third side.
4. Through two opposite sides of a parallelogram a number of planes are passed. Prove that all the intersections of these planes are parallel.
5. Prove: If a line is parallel to one of two parallel planes, it is parallel to the other. *Hint:* Pass a plane through the line intersecting the parallel planes. Prove directly.
6. Prove: If a plane is parallel to one of two parallel lines, it is parallel to the other.
7. Prove: If a plane is perpendicular to one of two perpendicular lines, it is parallel to the other.
8. Construct a line parallel to a given plane through an external point.

9. Construct a plane parallel to a given line and passing through two given points.

10. Construct two parallel planes, one through each of two skew lines.

SUPERIOR WORK

S 1. *ABCD* is a broken line not all in one plane. Prove that the midpoints of the three line segments determine a plane parallel to *AC* and *BD*.

S 2. Through a given line construct a plane that will intersect two given nonparallel planes in parallel lines.

S 3. Given three lines of which no two are in the same plane. Construct a fourth line that will intersect two of the given lines and be parallel to the third.

S 4. Prove: If two lines are parallel, they are parallel to the intersection of two planes each containing one of the lines.

S 5. In Exercise 8 on page 16, you proved that if three planes meet and two of the intersections meet, all three intersections meet in a point. Now prove that if two of the intersections are parallel, all three intersections are parallel.

S 6. Through a given point on a given line construct a line that will be perpendicular to the given line and parallel to a given plane.

Intersecting Planes

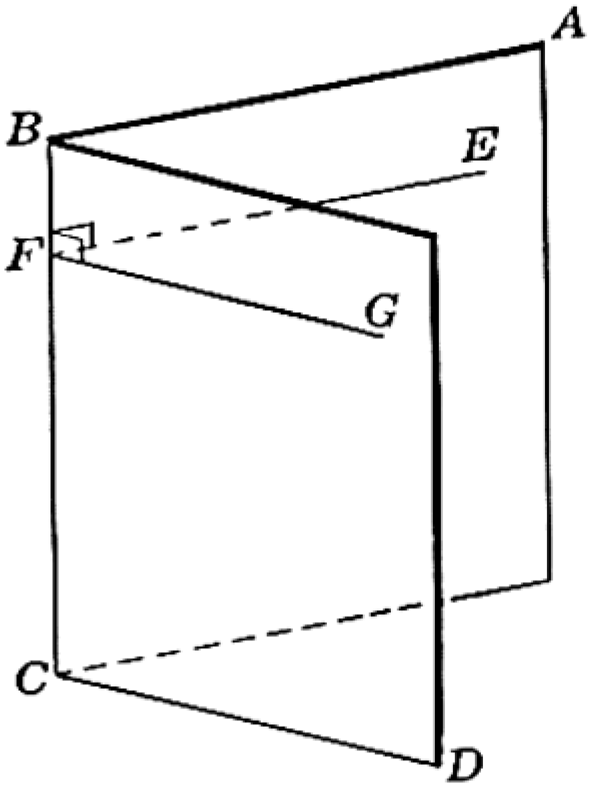
PERPENDICULAR PLANES

71. To understand the idea of perpendicular planes, we must first study the angle between two planes. Just as a point is said to divide a line into two half-lines or rays, so a line divides a plane into two half-planes. In a plane, an angle is formed by two rays from a common point. In space, the figure formed by two half-planes from a common line is called a *dihedral angle*.

72. Definition. *A dihedral angle is a figure formed by two half-planes from a common line called the edge.*

73. The dihedral angle in the figure at the top of page 51 is formed by the half-planes *AC* and *BD* from the common edge *BC*. It may be read as dihedral $\angle A\text{–}BC\text{–}D$. The middle letters denote the edge and the two end letters designate a point in each of the planes, called the *faces.* If the edge belongs to only one angle, as in the figure, the angle may be read simply as dihedral $\angle BC$.

In the same figure, EF is in the face AC and is perpendicular to the edge BC at F; also GF in the face BD is perpendicular to BC at F. The angle EFG is called the *plane angle* of the dihedral angle.



74. Definition. *A plane angle of a dihedral angle is an angle formed by two lines, one in each face, perpendicular to the edge of the dihedral angle at the same point.*

75. Definition. *A right section of a dihedral angle is the intersection of the dihedral angle with a plane perpendicular to the edge of the dihedral angle.*

Why is a right section of a dihedral angle a plane angle of the dihedral angle? Why are all the plane angles of the same dihedral angle equal?

76. Corollary. All the plane angles of the same dihedral angle are equal.

77. Definition. *The measure of a dihedral angle is any one of its plane angles.*

78. Corollary. If two dihedral angles are equal, their plane angles are equal; and conversely.

The various kinds of dihedral angles are usually defined in terms of their plane angles; that is, dihedral angles are classified according to their right sections.

79. Definitions. *A dihedral angle is right, acute, or obtuse according as its plane angle is right, acute, or obtuse.*

Two dihedral angles are adjacent, vertical, complementary, or supplementary according as their plane angles are adjacent, vertical, complementary, or supplementary.

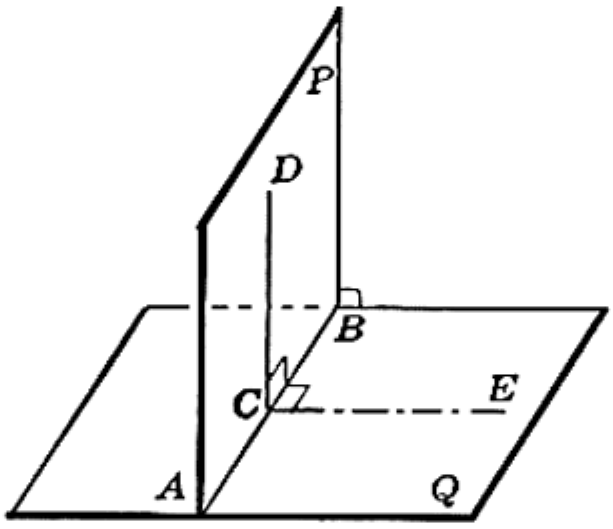
Two planes are perpendicular if they form right dihedral angles.

80. Corollary. Complements or supplements of equal dihedral angles are equal.

81. Corollary. Vertical dihedral angles are equal.

Based on perpendicular planes, the following theorem gives us another method of proving a line perpendicular to a plane.

82. THEOREM. If two planes are perpendicular to each other, a line in one of them perpendicular to their intersection is perpendicular to the other.



GIVEN: Planes P and Q intersect in AB.
$P \perp Q$.
In P, $CD \perp AB$.

PROVE: $CD \perp Q$.

CONS.: In Q draw $CE \perp AB$.

PROOF:

Statements	*Reasons*
1. $\angle DCE$ is a plane $\angle$ of dihed. $\angle P\text{–}AB\text{–}Q$.	1. Def. plane $\angle$ of dihed. $\angle$. (§ 74)
2. $P \perp Q$.	2. Given.
3. $\angle P\text{–}AB\text{–}Q$ is a rt. dihed. $\angle$.	3. Def. $\perp$ planes. (§ 79)
4. $\angle DCE$ is a rt. $\angle$.	4. Def. rt. dihed. $\angle$. (§ 79)
5. $CD \perp CE$.	5. Def. $\perp$ lines. (§ 378)
6. $CD \perp AB$.	6. Given.
7. $CD \perp Q$.	7. A line $\perp$ to two intersecting lines is $\perp$ to plane of these lines. (§ 30)

83. Corollary 1. If two planes are perpendicular to each other, a line through any given point in the first plane perpendicular to the second plane lies in the first plane.

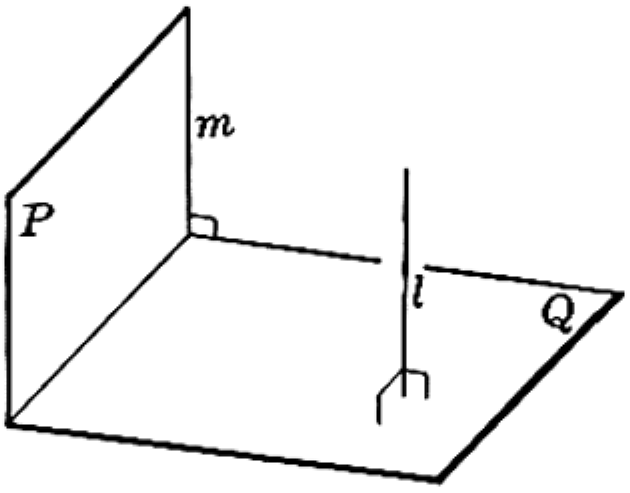
Assuming that the line does not lie in the first plane, draw another line in the first plane through the given point perpendicular to the intersection.

Draw a figure and write out the indirect proof.

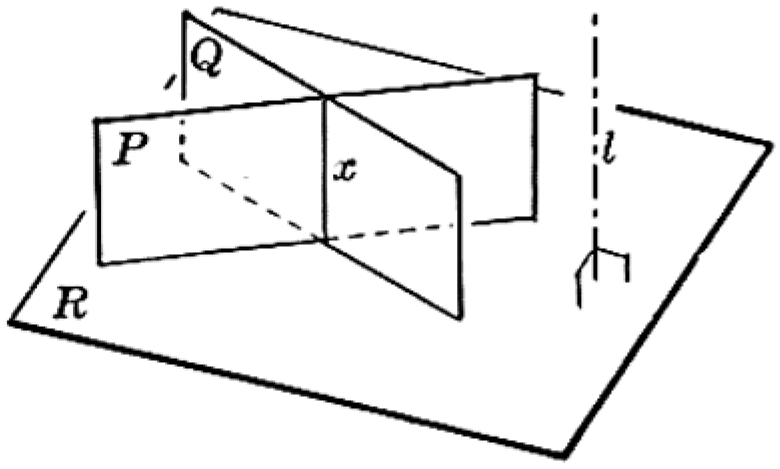
84. Corollary 2. If a line is perpendicular to one of two perpendicular planes, it is parallel to the other.

If plane P and line l could meet, then by § 83, l would lie in P.

In the figure if plane P can rotate about a line m while remaining perpendicular to plane Q, how is m related to Q? The following theorem answers this question.



85. THEOREM. If each of two intersecting planes is perpendicular to a third plane, the intersection is also perpendicular to the third plane.



GIVEN: Plane $P \perp$ plane R.
Plane $Q \perp$ plane R.
P and Q intersect in x.

PROVE: $x \perp R$.

CONS.: Draw any line $l \perp R$.

PROOF:

Statements	*Reasons*
1. $l \parallel P$, $l \parallel Q$.	1. If a line $\perp$ to one of 2 $\perp$ planes, it is $\parallel$ to the other. (§ 84)
2. $l \parallel x$.	2. If a line is $\parallel$ to 2 intersecting planes, it is $\parallel$ to their intersection. (§ 65)
3. $x \perp R$.	3. If one of 2 $\parallel$ lines is $\perp$ to a plane, the other is also $\perp$ to the plane. (§ 49)

EXERCISES

1. a. Draw a line perpendicular to the edge of a sheet of paper. Fold the paper along this line to form a model of a dihedral angle.

b. What is the name of the angle formed by the two parts of the folded edge?

c. From a common point in the edge of your dihedral angle, draw a line in each face. Is the angle between these lines necessarily equal to the plane angle of the dihedral angle?

d. By repeating (c) with various pairs of lines, determine the largest and the smallest angles that may be formed by a plane section of a dihedral angle.

2. a. Draw a cube and letter the vertices.
   b. How many dihedral angles are in a cube? Name some of them.
   c. Assuming, as we did before, that the faces of a cube are squares, prove that any two intersecting edges form a plane angle of a dihedral angle.

3. Prove: If a line in one face of a dihedral angle is parallel to a line in the other face, the plane of the lines is parallel to the edge of the dihedral angle.

4. Given a dihedral angle, construct a plane that will bisect it.

5. Prove that the planes that bisect two adjacent supplementary dihedral angles are perpendicular.

6. Prove: **a.** If two parallel planes are cut by a third plane, the corresponding dihedral angles are equal. Is the converse valid?
   b. A plane perpendicular to one of two parallel planes is perpendicular to the other. Is the converse valid?

7. Prove: If two planes are both perpendicular to each of two intersecting planes, they are parallel.

8. Prove: If a plane is parallel to the edge of a dihedral angle but not to either of its faces, it intersects the faces in parallel lines.

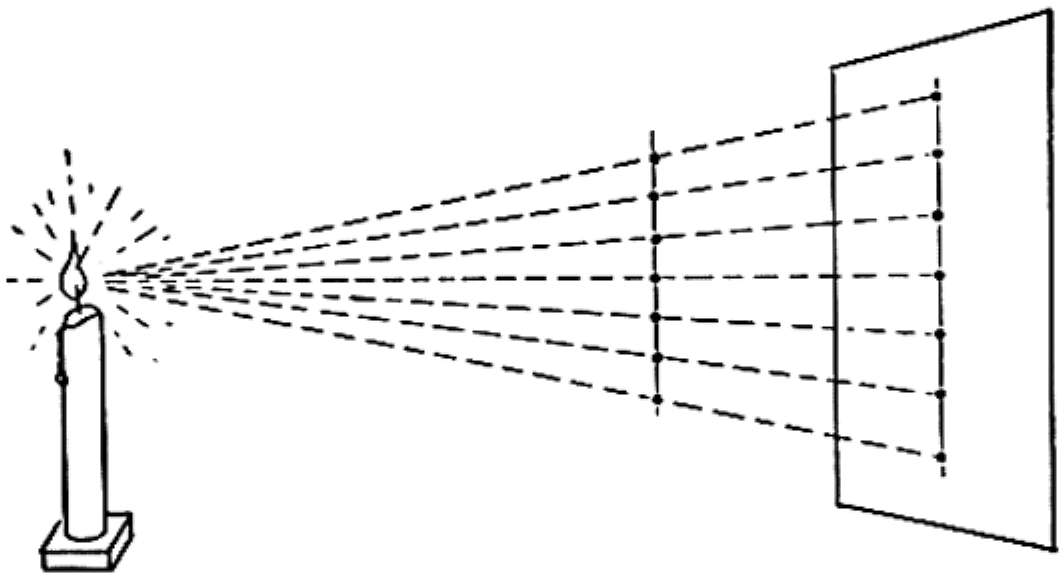
SUPERIOR WORK

S 1. a. What would have to be known about the intersections to make the converse of Exercise 6 (a) valid? Assuming this condition, prove the converse.
   b. State the converse of the proposition in Exercise 6 (b), incorporating the condition necessary to make it valid.

S 2. a. If a plane is not perpendicular to the edge of a dihedral angle, it is not perpendicular to both faces of the dihedral angle. Prove this indirectly.
   b. State the opposite-converse of the proposition in (a) and prove it directly.
   c. Using the axiom that a proposition and its opposite-converse are equally valid (page 19, S 1), we may use the proof in (a) or the proof in (b). Which proof do you think is simpler? Why?

PROJECTIONS

86. In order to represent and to study the relationships between figures in space, especially those that do not intersect, we draw them in a plane by means of a device known as *projection*. By drawing certain lines, called *projecting lines*, through the figure and the plane, we associate the points in the figure with those in the plane. There are a number of ways of drawing the projecting lines through the figure, each way resulting in a different projection on the plane. Thus, one way is to draw all the projecting lines through a point like the rays of light from a candle projecting the shadow of an object on a wall. This is called *central* projection and is used to make certain kinds of maps. If the projecting lines are perpendicular to the plane of projection, the result is called an *orthogonal* projection. Our projections will all be orthogonal, and we shall refer to them merely as projections. A draftsman's projections are largely of this kind.



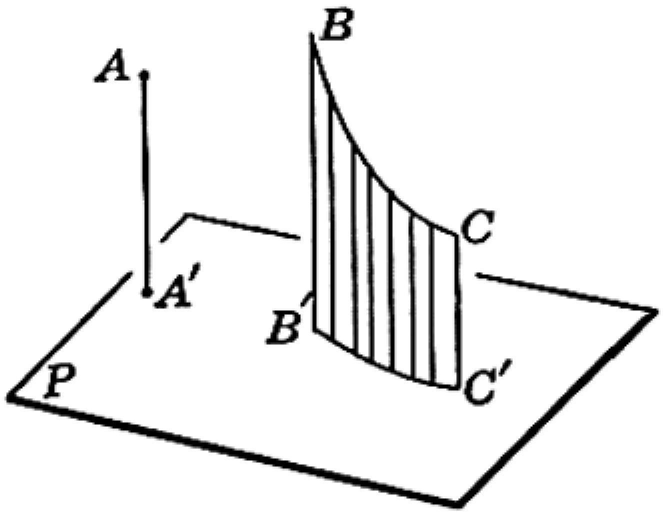
87. Definitions. *The projection of a point on a plane is the foot of the perpendicular from the point to the plane.*

The perpendicular is called the projecting line; the plane is called the plane of projection.

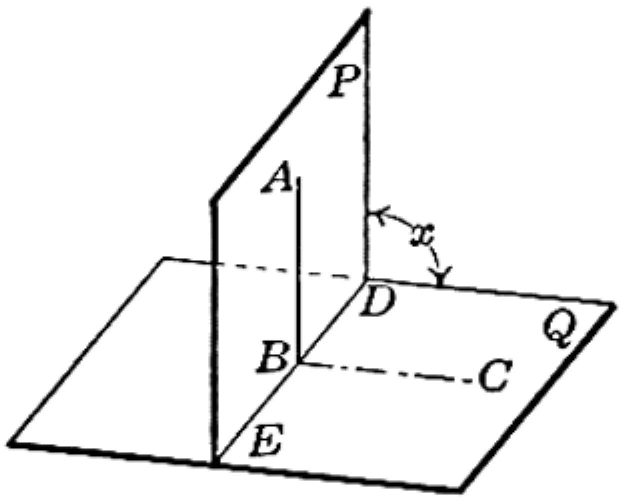
The projection of a line (straight or curved) on a plane is the locus of the projections on the plane of all the points of the line.

In the figure, A' is the projection of A on plane P; $B'C'$ is the projection of BC.

We would like to prove that a straight line can be projected on a plane by passing another plane through the line and making it perpendicular to the plane of projection. To do this, we must first prove a theorem giving the relation between two planes of which one contains a line perpendicular to the other.



88. THEOREM. **If a line is perpendicular to a given plane, every plane which contains this line is perpendicular to the given plane.**



GIVEN: $AB \perp$ plane Q.
Plane P contains AB.
P and Q intersect in DE.

PROVE: $P \perp Q$.

CONS.: In Q draw $BC \perp DE$.

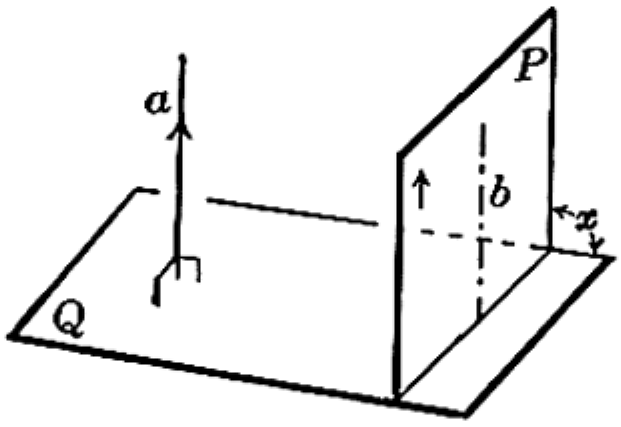
Write out the proof. *Hint:* Prove that ABC is a plane angle of the dihed. $\angle P\text{–}DE\text{–}Q$.

89. Corollary 1. Through a given line not perpendicular to a given plane one, and only one, plane can be perpendicular to the given plane.

From any point in the given line draw a line perpendicular to the given plane. Then pass a plane through the two lines.

Prove that your construction is unique by showing that, if there were two such planes, they would intersect in a line perpendicular to the given plane (§ 85).

90. Corollary 2. If a plane is parallel to a line, then another plane perpendicular to the line is also perpendicular to the first plane.



GIVEN: Plane $P \parallel a$.
$a \perp$ plane Q.

PROVE: $P \perp Q$.

CONS.: Through a draw a plane cutting P in b.

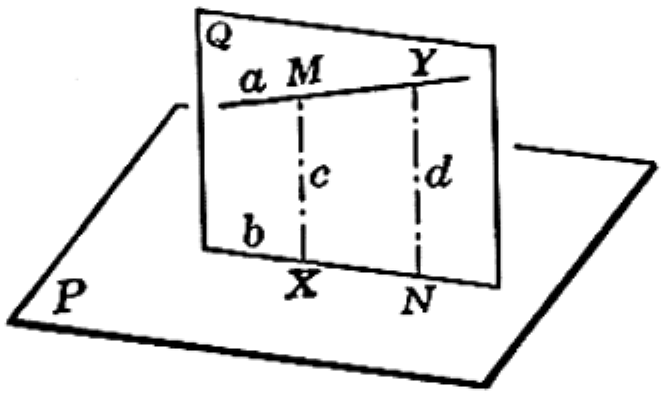
PROOF:

1. $a \parallel b$.
2. $b \perp Q$.
3. $P \perp Q$.

Supply the reason for each statement.

We are now ready to prove that if a line is not perpendicular to a plane, (1) its projection on the plane is a straight line and (2) the projection is the intersection of a plane through the projected line perpendicular to the plane of projection.

91. THEOREM. The projection on a plane of a straight line not perpendicular to the plane is a straight line which is the intersection of the given plane with a plane perpendicular to it and containing the given line.



GIVEN: Line a not $\perp$ plane P.
Plane Q contains a and intersects P in b.
$Q \perp P$.

PROVE: b is the projection of a on P.

PLAN OF PROOF: Since the projection of a is a locus, we must prove:

a. The projection of any point belonging to a is on b.
b. Any point on b is the projection of some point on a.

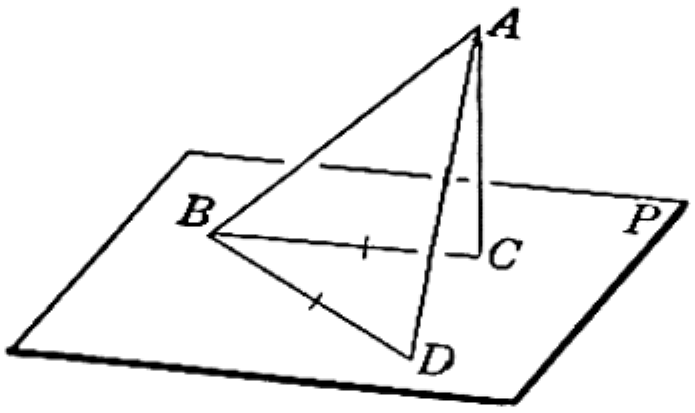
PROOF (Optional):

Statements	*Reasons*
a. Take any pt. M on a and prove that its projection is on b.	
1. Through M draw $c \perp P$.	1. § 36.
2. c is in Q.	2. § 83.
3. c intersects b in some pt. X.	3. § 14.
4. X is the projection of M on P.	4. Def. projection of pt. on plane. (§ 87)
b. Take any pt. N on b and prove that it is the projection of some pt. on a.	
5. Through N draw $d \perp P$.	5. § 36.
6. d is in Q.	6. § 83.
7. d intersects a in some pt. Y.	7. § 379.
8. N is the projection of Y on P.	8. Same as 4.

92. Corollary. The projection on a plane of a line segment is the part of the projection of the line between the projections of the end points of the line segment.

93. Definition. *A plane that passes through a line and is perpendicular to a given plane is called the projecting plane of the line on the given plane.*

94. THEOREM (Optional). **The acute angle which a line makes with its projection on a plane is the smallest angle which it makes with any line in the plane.**



GIVEN: BC the projection of AB on plane P.
BD any other line in P through B.

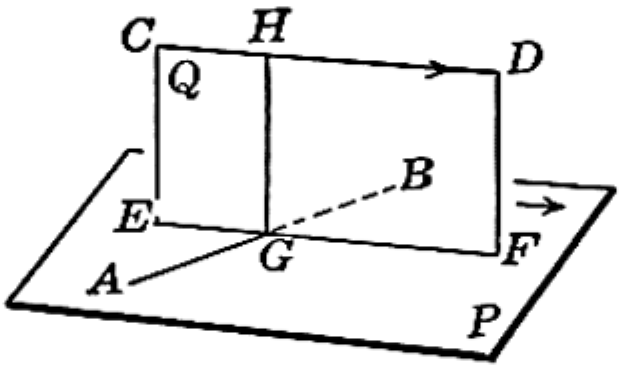
PROVE: $\angle ABC < \angle ABD$.

CONS.: Lay off $BD = BC$, and draw AD.

Write out the proof. (Use § 40 and § 472.)

95. Definition. *The acute angle which a line makes with its projection on a plane is called the angle which the line makes with the plane, or its inclination to the plane.*

96. THEOREM. Any two skew lines have a common perpendicular.



GIVEN: Skew lines AB and CD.

PROVE: AB and CD have a common perpendicular.

PROOF:

Statements	*Reasons*
1. Through AB draw plane $P \parallel CD$.	1. § 69.
2. Project CD on P, letting the projecting plane Q intersect P in EF.	2. § 91.
3. $EF \parallel CD$.	3. § 64.
4. $EF \nparallel AB$.	4. § 51.
5. EF intersects AB in some pt. G.	5. Def. $\parallel$ lines.
6. In Q draw $GH \perp EF$.	6. § 539.
7. $GH \perp P$.	7. § 82.
8. $GH \perp AB$.	8. Def. $\perp$ to plane.
9. $GH \perp CD$.	9. § 462.

EXERCISES

1. a. What is the location of two points that have the same projection on a plane?

b. Three given points are not collinear, but their projections on a plane are. How is the plane of projection related to the plane of the given points?

c. The projections on a plane of two given skew lines are parallel. What do you know about two parallel planes that contain the skew lines? (Page 50, Exercise 10)

d. Describe the position of a line segment relative to a plane if its projection on the plane equals the line segment; equals half the line segment; is a point.

e. How does the obtuse angle which a line makes with its projection on a plane compare with the obtuse angle it makes with any other line in the plane? Why?

f. Why does a swinging door remain perpendicular to the floor?

2. A plumb line is a line that points directly to the center of the earth.

a. Are all plumb lines parallel?

b. Define a horizontal plane in terms of a plumb line. Are all horizontal planes parallel? Explain.

c. Define a vertical plane in terms of a plumb line. Are all vertical planes parallel? Explain.

3. Prove: If a plane is perpendicular to a line in another plane, it is perpendicular to the other plane.

4. Through a given point within a dihedral angle construct a plane that is perpendicular to each face of the angle.

5. Prove: If the projections of two parallel lines on the same plane are not the same line, they are parallel.

6. Prove: The projection on a plane of a parallelogram is either a parallelogram or a straight line.

7. Prove: The projections on a plane of two parallel line segments are proportional to the line segments.

8. Prove: If a line segment l makes an angle θ with a plane, its projection p on the plane is given by the formula $p = l \cos \theta$ (§ 430).

9. Prove: The legs of an isosceles triangle make equal angles with any plane that contains the base.

10. Prove: If two line segments from a point to a plane make equal angles with the plane, they are equal.

11. Prove: If a line intersects two parallel planes, it makes equal alternate interior angles with the planes.

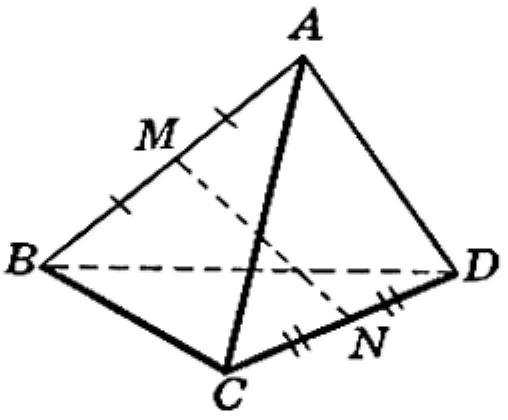
12. a. Given the projections of a point on two intersecting planes, how do you locate the projected point?

b. Prove: If the projections of a figure on two intersecting planes are straight lines, then the projected figure is a straight line.

13. A dihedral angle is 100°. Find the number of degrees in the angle formed by the perpendiculars to the faces from a point within the dihedral angle; from a point outside.

14. Through a given point construct a plane perpendicular to a given plane and parallel to a given line. When is the construction impossible?

15. The figure shown is a solid enclosed by four equilateral triangles. M and N are the midpoints of the opposite sides AB and CD. Prove that MN is the common perpendicular to the skew lines AB and CD. *Hint:* Apply the two-pt. theorem (§ 449) to M and N.



SUPERIOR WORK

S 1. Three concurrent lines are perpendicular to the three faces of two adjacent dihedral angles. Prove that the three lines are in one plane.

S 2. Prove: If one side of a right angle is parallel to a plane, the projection of the right angle on the plane is also a right angle.

S 3. Prove: If a line is parallel to one of two perpendicular planes, its projection on the other plane is parallel to the edge of the dihedral angle formed by the planes.

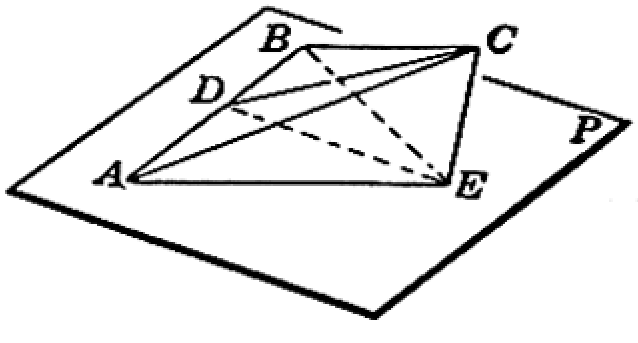
S 4. Prove: If a line is perpendicular to one of two intersecting planes, its projection on the other plane is perpendicular to the edge of the dihedral angle formed by the planes.

S 5. Prove the converse of Exercise 11, page 59.

S 6. If the faces of two dihedral angles are respectively perpendicular and their edges are parallel, the dihedral angles are either equal or supplementary.

S 7. If the four vertices of a parallelogram and the intersection of the diagonals are projected on a plane, the projecting line of the intersection of the diagonals is the average of the other four projecting lines.

S 8. In the figure the equilateral triangle ABC has one side AB in plane P; the plane ABC makes an angle of 30° with P; $\triangle ABE$ is the projection on P of $\triangle ABC$; $AB = 12$ units. Find the perimeter and the area of $\triangle ABE$.



Help: In a complicated figure like this it is helpful to draw some of the sections in their true shape:

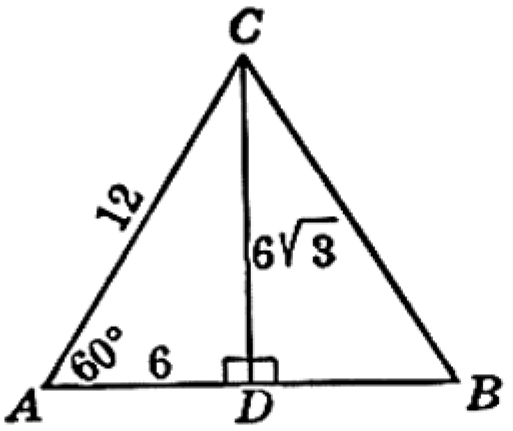


Fig. 1

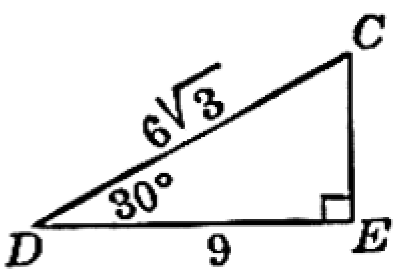


Fig. 2

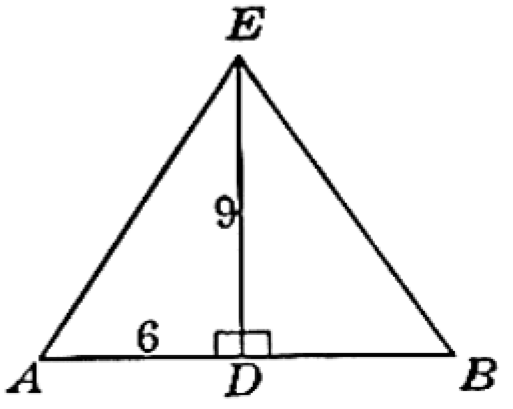


Fig. 3

In Fig. 1, we find $CD = 6\sqrt{3}$; in Fig. 2, $DE = 9$.

S 9. One side of an equilateral triangle is parallel to a plane P, and the plane of the triangle makes an angle θ with P. Prove that the area of the projection of the triangle on P equals the product of the area of the original triangle and the cosine of θ. (See Exercise 8.)

Note: If S is the area of any plane figure and S' is the area of the projection of S on another plane, it can be proved that $S' = S \cos \theta$, where θ is the angle between the planes containing S and S'.

S 10. a. Prove indirectly that there is only one common perpendicular between two given skew lines.

Outline of Proof: Copy the figure in § 96. Suppose that in your figure there is another common perpendicular to AB and CD, cutting AB in X, and CD in Y. Draw XY.

Through X draw $XM \parallel CD$; through Y draw $YN \perp P$ at N.

Prove that YX and YN would be two perpendiculars from the same point Y to P. Write out the proof.

b. Prove that the common perpendicular between two skew lines is the shortest line between them.

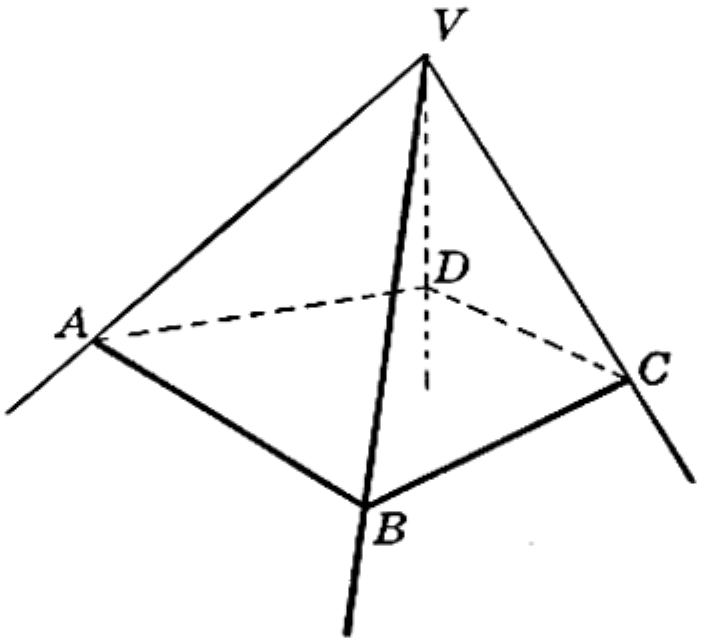
Outline of Proof: In your figure for (a), prove that the common perpendicular HG is less than any other line such as YX. $HG = YN < YX$ (§ 40). Write out the proof.

FIGURES MADE BY THREE OR MORE PLANES

Up to now we have been concerned chiefly with figures involving two planes. In this lesson we shall begin the study of figures made up of three or more planes. If three or more planes meet in a point, they form a figure called a *polyhedral angle.* We shall define a polyhedral angle in terms of motion as follows.

97. Definitions. *A polyhedral angle is a figure that is generated by the rotation of a ray about its initial point while intersecting a polygon in another plane.*

The initial point of the ray is the vertex, the portions of the planes generated are the faces, the intersections of the faces are the edges, the *angles between the successive edges are the face angles, and the angles between the successive faces are the dihedral angles of the polyhedral angle.*

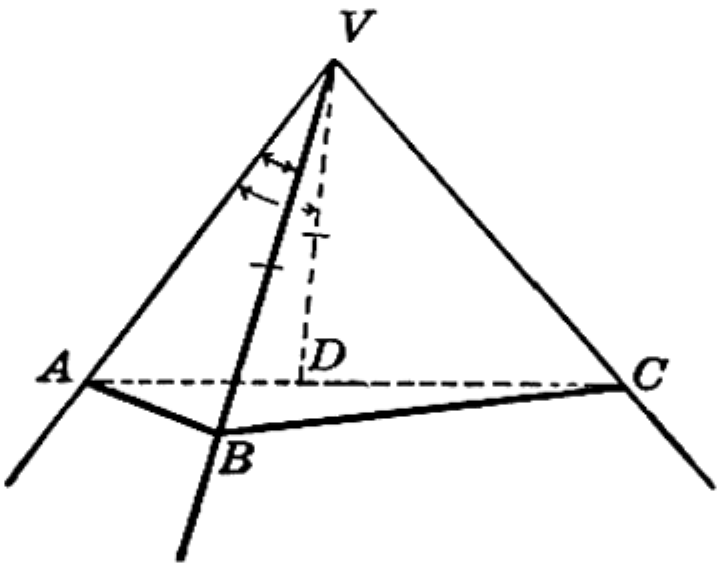


The polyhedral angle in the figure is read "polyhedral ∠ *V–ABCD*" or, if there is only one polyhedral angle at the vertex, simply "polyhedral ∠ *V*." *V* is the vertex; *VAB*, *VBC*, . . . are the faces; *VA*, *VB*, . . . are the edges; ∠*AVB*, ∠*BVC*, . . . are the face angles; angles *VA*, *VB*, . . . are the dihedral angles of the polyhedral angle.

98. Definition. *A polyhedral angle is a trihedral angle, a tetrahedral angle, a pentahedral angle, etc., according as it has three, four, five faces, etc.*

If the polygon intersected by the rotating ray is convex, the polyhedral angle is said to be *convex*. We shall study only convex polyhedral angles. In this lesson we shall study the face angles of a polyhedral angle.

99. THEOREM. The sum of two face angles of a trihedral angle is greater than the third face angle.



Given: Trihed. ∠*V–ABC*, with ∠*AVC* the largest face angle.

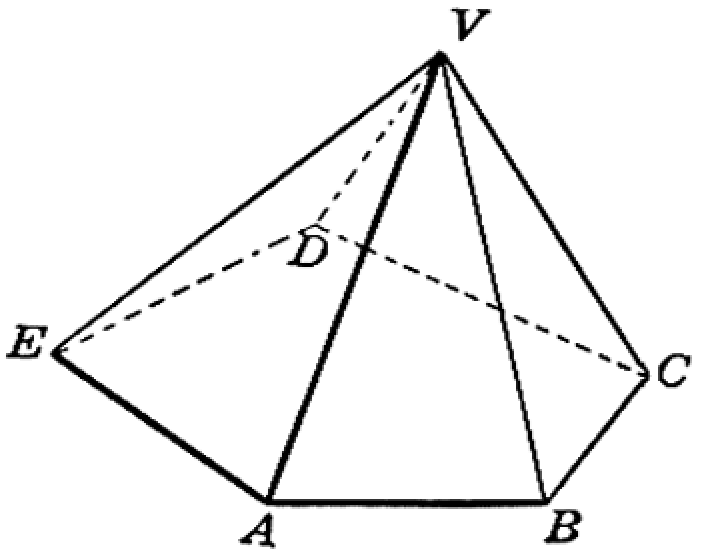
Prove: ∠*AVB* + ∠*BVC* > ∠*AVC*.

Cons.: Draw *AC*. In face *AVC* draw *VD* making ∠*AVD* = ∠*AVB*.
Lay off *VB* = *VD*. Draw *AB* and *BC*.

Proof:

1. $\triangle AVB \cong \triangle AVD$.
2. $AB = AD$.
3. $AB + BC > AC$.
4. $BC > DC$.
5. $\angle BVC > \angle DVC$. (§472)
6. Since $\angle AVB = \angle AVD$,
7. $\angle AVB + \angle BVC > \angle AVC$.

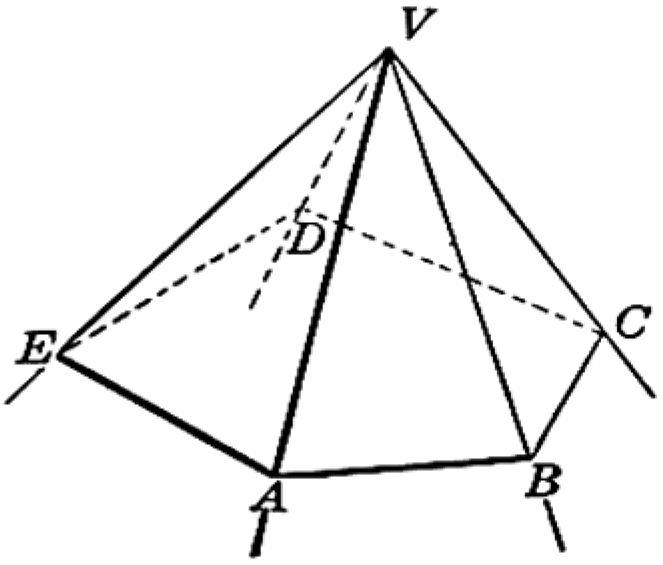
Supply the reason for each step of the proof.



In the figure imagine that the point V moves farther away from the plane of the polygon $ABCDE$. Do you think that the sum of the face angles at V would increase, decrease, or remain the same? How would this sum be affected if V came closer to the polygon? If you decide that the sum of the angles is a variable quantity, can you tell between what limits it varies? (Cf. page 38, Exercise 19.)

Your intuitive conclusions are proved in the following theorem.

100. THEOREM. The sum of the face angles of any polyhedral angle is less than two straight angles.



GIVEN: Polyhed. $\angle V\text{-}ABCD\ldots$ having n faces.
$S = \angle AVB + \angle BVC + \angle CVD + \cdots$

PROVE: $S < 2$ st. ∡.

CONS.: Draw a plane intersecting all the edges and forming polygon $ABCD \ldots$.

Let T = the sum of the angles of the triangles except those at V, and let P = the sum of the angles of the polygon; that is,

$$T = \angle VAE + \angle VAB + \angle VBA + \cdots$$

and

$$P = \angle EAB + \angle ABC + \angle BCD + \cdots$$

PROOF: *Statements*	*Reasons*
1. $\angle VAE + \angle VAB > \angle EAB$. $\angle VBA + \angle VBC > \angle ABC$. . . .	1. The sum of 2 face ∡ of a trihed. ∠ > third face ∠. (§ 99)
2. $T > P$.	2. Unequals added to unequals, sums unequal in same order. (§ 442)
3. $P = (n - 2)$ st. ∡.	3. Sum of ∡ of polygon = $(n - 2)$ st. ∡. (§ 492)
4. $T > (n - 2)$ st. ∡.	4. Subs. Ax.
5. $S + T = n$ st. ∡.	5. Sum of ∡ of △ = 1 st. ∠.
6. $S < 2$ st. ∡.	6. Unequals subtracted from equals, remainders ≠ in opposite order. (§ 444)

EXERCISES

1. Is it possible for a trihedral angle to have the following face angles?

a. 50°, 100°, 60°	**d.** 60°, 90°, 150°
b. 50°, 100°, 140°	**e.** 170°, 170°, 30°
c. 50°, 100°, 40°	**f.** 90°, 120°, 40°

2. Prove that the difference between two face angles of a trihedral angle is less than the third face angle.

3. In each of the following, two face angles of a trihedral angle are given. Determine between what limits the third face angle must be.

a. 70°, 50°	**d.** 120°, 120°
b. 120°, 50°	**e.** 170°, 10°
c. 110°, 80°	**f.** 90°, 90°

4. Prove that the sum of three face angles of a tetrahedral angle is greater than the fourth face angle. *Hint:* Draw a plane through two opposite edges of the tetrahedral angle, dividing it into two trihedral angles.

5. Determine whether or not each of the following combinations can be the face angles of a polyhedral angle. (Consider the preceding exercise.)

a. 90°, 70°, 50°, 100°, 45°	**c.** 90°, 170°, 40°, 30°
b. 80°, 120°, 60°, 100°	**d.** 175°, 60°, 60°, 60°

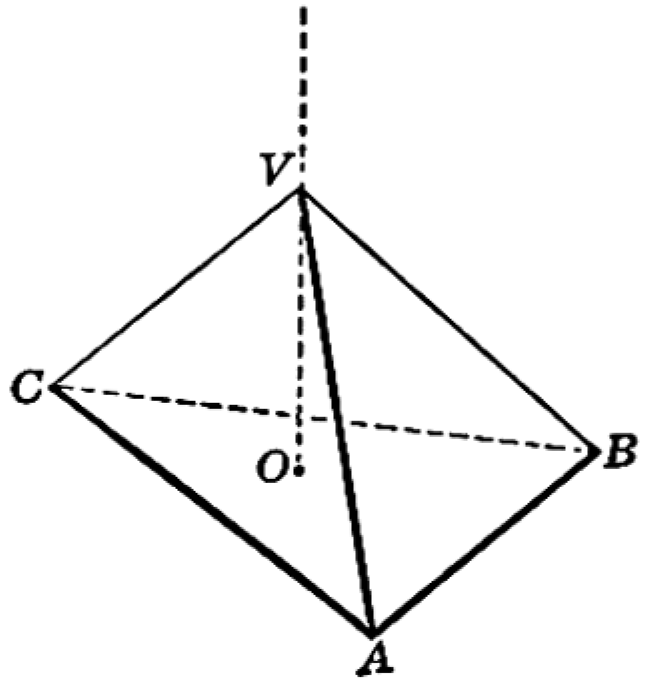
6. **a.** Can a polyhedral angle be formed by fitting together three equilateral triangles? four? five? six? seven?

   b. Can a polyhedral angle be formed by fitting together regular hexagons?

7. How many face angles can a polyhedral angle have if each face angle is the angle of an equilateral triangle? of a square? of a regular pentagon? of a regular hexagon? of a regular heptagon?

8. **a.** In the figure, VO is perpendicular to the plane of $\triangle ABC$ at its center O. Imagine that V moves up indefinitely high along OV. The edges VA, VB, and VC approach a position where they are perpendicular to plane ABC.

   What angles in the figure become the plane angles of the dihedral angles VA, VB, and VC? In that limiting position what is the sum of the dihedral angles?



   b. If, in the same figure, V moves down and approaches O, do the dihedral angles increase? What is the upper limit of each dihedral angle? of their sum?

c. Between what limits is the sum of the dihedral angles of a trihedral angle?

9. Using your conclusion in Exercise 8(c), determine whether or not each of the following combinations can be the dihedral angles of a trihedral angle:

a. 70°, 60°, 50°
c. 100°, 40°, 30°
b. 40°, 130°, 20°
d. 170°, 170°, 170°

10. Using a diagram similar to the one in Exercise 8, determine the upper and lower limits of the sum of the n dihedral angles of a polyhedral angle.

SUPERIOR WORK

S 1. Prove that the sum of the angles of a skew quadrilateral is less than 360°.

S 2. In a trihedral angle each of three planes is determined by one edge of the trihedral angle and the bisector of the opposite face angle. Prove that the three planes meet in one line. *Hint:* Pass a plane cutting the three edges of the trihedral angle at equal distances from the vertex.

S 3. If the larger of two face angles of a trihedral angle has m degrees and the smaller n degrees, prove that the number of degrees x in the third face angle is limited as follows:

$$x > m - n, \qquad x < m + n, \qquad \text{and} \qquad x < 360° - (m + n)$$

Apply this test to Exercises 1 and 3 above.

POLYHEDRONS

We shall now take up solids enclosed by planes. We recall from page 5 that a geometric solid is a portion of space. What is the least number of planes that can enclose a portion of space?

101. Definitions. *A polyhedron is a solid bounded by plane polygons.*

The bounding polygons are the faces, the intersections of the faces are the edges, and the points where three or more faces meet are the vertices of the polyhedron.

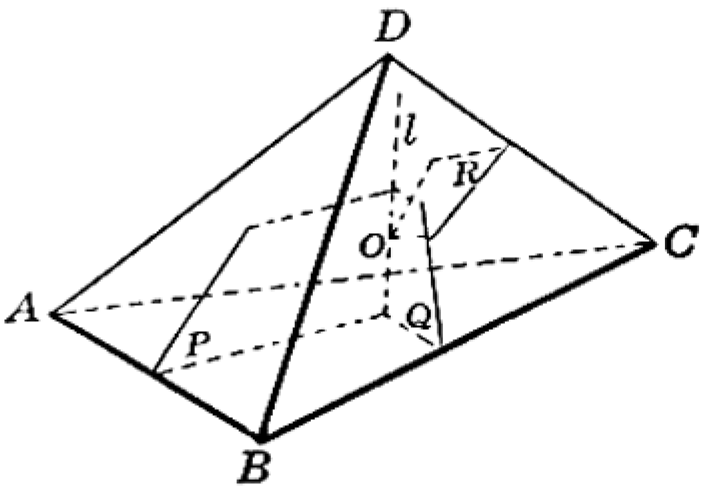
A line segment joining any two vertices not in the same face is a diagonal of the polyhedron.

A polyhedron is a tetrahedron, a hexahedron, an octahedron, a dodecahedron, or an icosahedron according as it has four, six, eight, twelve, or twenty faces.

If every section made by a plane intersecting a polyhedron is a convex polygon, the polyhedron is said to be convex. We shall study only convex polyhedrons.

A tetrahedron in general space is analogous to a triangle in a plane. A tetrahedron encloses a portion of space with the minimum number of planes; a triangle encloses a portion of a plane with the minimum number of straight lines. The faces and the dihedral angles of a tetrahedron are analogous to the sides and the angles of a triangle respectively. In every triangle there is a point equidistant from the three vertices and a point equidistant from the three sides. We shall prove that in every tetrahedron there is a point equidistant from the four vertices and a point equidistant from the four faces.

102. THEOREM. There is a point that is equidistant from the four vertices of a tetrahedron.



GIVEN: Tetrahedron $ABCD$.

PROVE: There is a pt. O equidistant from A, B, C, and D.

CONS.: Draw planes P, Q, and R, the ⊥ bisectors of AB, BC, and CD respectively.

PLAN OF PROOF:

P and Q intersect in some line l, and l intersects R in O.

Since O is in P, Q, and R, it is equidistant from A, B, C, and D.

PROOF (Optional):

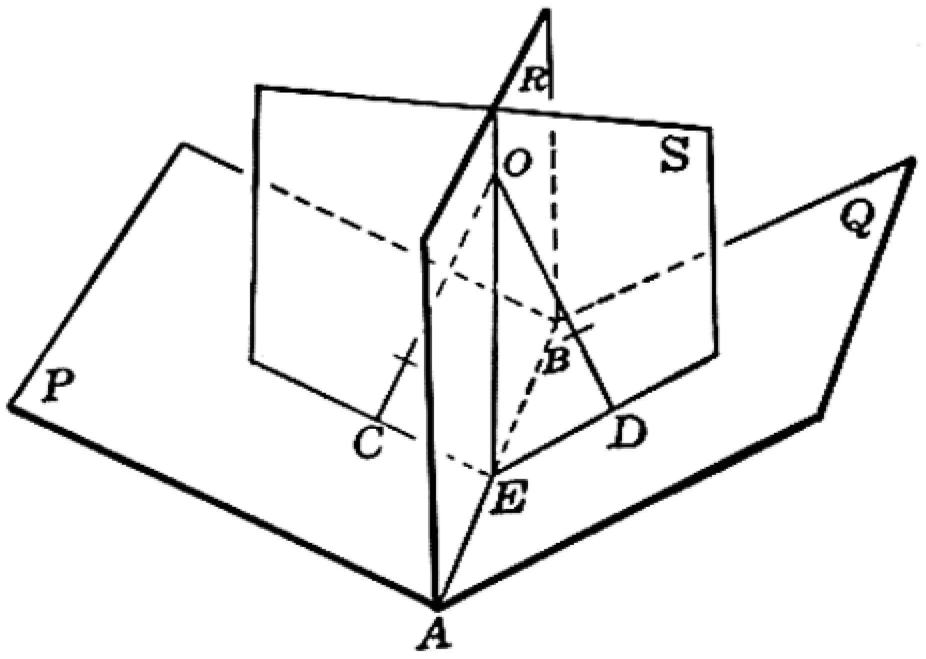
Statements	*Reasons*
1. P and Q meet in some line l.	1. If 2 planes are ⊥ to 2 intersecting lines, they are ∦. (§ 59)
2. P and Q are ⊥ plane ABC.	2. A plane containing a line ⊥ to another plane is itself ⊥ to that plane. (§ 88)
3. l ⊥ plane ABC.	3. If 2 intersecting planes are ⊥ to a third plane, their intersection is ⊥ to third plane. (§ 85)
4. Suppose $l \parallel R$.	4.
(*a*) Plane ABC ⊥ R.	(*a*) If a plane is ∥ to a line, then another plane ⊥ to the line is also ⊥ to the first plane. (§ 90)

(*b*) $CD \parallel$ plane ABC.	(*b*) If a line is $\perp$ to one of 2 $\perp$ planes, it is $\parallel$ to the other. (§ 84)
(*c*) This is impossible.	(*c*) Def. line $\parallel$ to plane.
5. Hence l meets R in some pt. O.	5. Def. line $\parallel$ to plane.
6. O, being in P, Q, and R, is equidistant from A, B, C, and D.	6. Every pt. in the plane that is the $\perp$ bisector of a line segment is equidistant from the ends of the line segment. (§ 45)

Why is there only one such point O?

To prove that there is a point equidistant from the faces of a tetrahedron, we shall first find the locus of a point equidistant from the faces of a dihedral angle.

103. THEOREM. A plane determined by the edge of a dihedral angle and a point equidistant from its faces bisects the dihedral angle.



Given: Dihed. $\angle P\text{–}AB\text{–}Q$.
$OC \perp$ plane P.
$OD \perp$ plane Q.
$OC = OD$.
Plane R passes through AB and O.

Prove: R bisects dihed. $\angle P\text{–}AB\text{–}Q$.

Cons.: Through OC and OD draw plane S, intersecting P in CE, Q in DE, and R in OE.

Proof:

Supply the reason for each statement.

1. $OC \perp P$, $OD \perp Q$.
2. $S \perp P$, $S \perp Q$.
3. $S \perp AB$.
4. CE, DE, and OE are $\perp AB$ at E.
5. $\measuredangle CEO$ and OED are the plane $\measuredangle$ of dihed. $\measuredangle P\text{–}AB\text{–}R$ and $R\text{–}AB\text{–}Q$ respectively.
6. $OC \perp CE$, $OD \perp DE$.
7. $OC = OD$.
8. $\angle CEO = \angle OED$.
9. Dihed. $\angle P\text{–}AB\text{–}R =$ dihed. $\angle R\text{–}AB\text{–}Q$.

104. Corollary. If a point is equidistant from the faces of a dihedral angle, it is in the plane that bisects the dihedral angle.

By proving the converse of this corollary, we shall be able to state that the bisecting plane of a dihedral angle is the locus of a point equidistant from the faces.

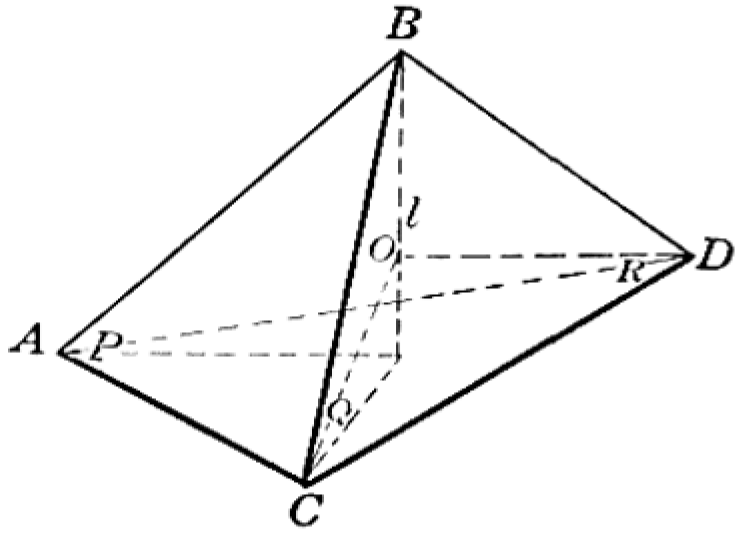
105. CONVERSE THEOREM. If a point is in the plane that bisects a dihedral angle, it is equidistant from the faces of the dihedral angle.

Draw a figure like the one in § 103 and write out the proof.

The following theorem summarizes § 104 and § 105.

106. LOCUS THEOREM. The locus of a point equidistant from the faces of a dihedral angle is the plane that bisects the dihedral angle.

107. THEOREM. There is a point that is equidistant from the four faces of a tetrahedron.



GIVEN: Tetrahedron $ABCD$.

PROVE: There is a pt. O equidistant from the four faces.

CONS.: Draw planes P, Q, and R to bisect dihed. $\measuredangle AB$, BC, and CD respectively.

PLAN OF PROOF:

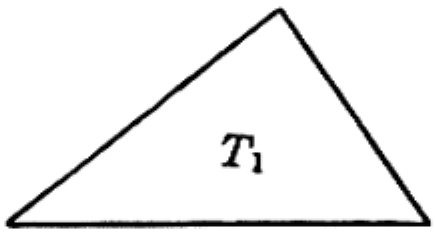
P and Q meet in some line l, and l intersects R and O.
Since O is in P, Q, and R, it is equidistant from the four faces.

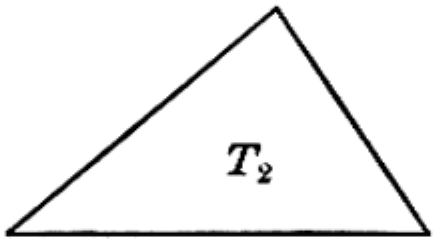
PROOF:

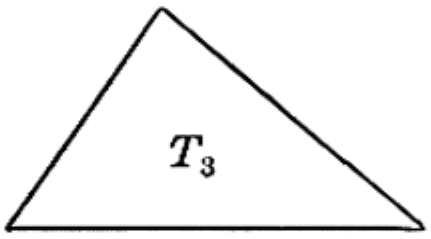
Statements	Reasons
1. P and Q are not $\parallel$, so they meet in some line l.	1. Def. $\parallel$ planes.
2. Since l extends from one face of dihed. $\angle CD$ to the other, it must intersect R, the bisector of dihed. $\angle CD$, in some pt. O.	2. Def. of bisector.
3. O, being in P, Q, and R, is equidistant from the four faces.	3. A pt. on the bisector of a dihed. $\angle$ is equidistant from the faces of dihed. $\angle$. (§ 105)

Why is there only one such point O?

108. Congruence and Symmetry. In the figure, triangles T_1 and T_2 agree in all their parts and also in the arrangement of these parts. We could fit these triangles together by merely shifting them on the paper. Triangles T_2 and T_3 also agree in all their parts, but these parts are arranged in the opposite order. By cutting out two triangles like T_2 and T_3, you can easily demonstrate that no amount of shifting on the surface of your desk can make these triangles fit together. This, however, caused us no concern in plane geometry because we could always turn triangle T_3 "out of the plane" of the paper and make it like the other two.

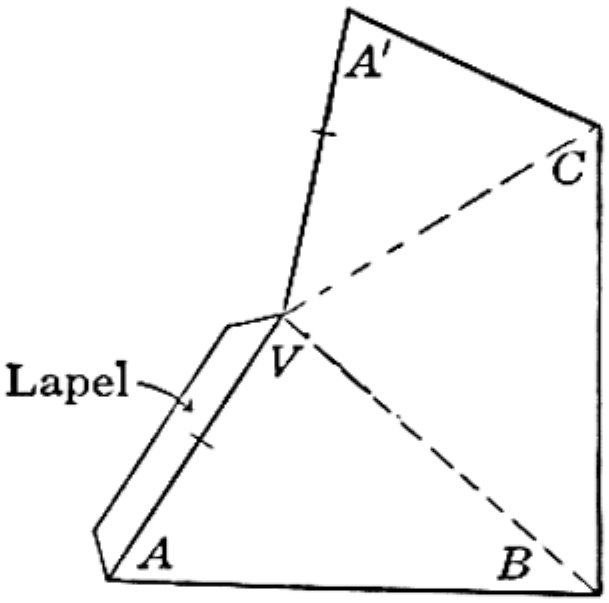






If two solid figures agree in all their parts which are arranged in the opposite order, we cannot lift one out of space and make it like the other. We therefore do not call such solids congruent; we say they are *symmetric*. A pair of gloves, and an object and its reflection in a mirror, are examples of symmetric solids.

By cutting three sheets of paper simultaneously, make three patterns like the one shown. Fold one pattern along the dotted lines until VA and VA' fit together. Paste the lapel so as to form a model of a trihedral angle. Do the same with your second pattern. Make a similar model from your third pattern but fold it in the opposite way from the other two.



Can you fit your first model into the second? Can you fit your third model into the second? What does this experiment show about two trihedral angles whose face angles are respectively equal? Can you make your third model like the other two by turning it "inside out"? Can you do the same with a pair of gloves?

109. Definitions. *Congruent solids are solids that have the parts of one equal respectively to the corresponding parts of the other and arranged in the same order.*

Symmetric solids are solids that have the parts of one equal respectively to the corresponding parts of the other but arranged in the opposite order.

110. Definition. *A regular polyhedron is one whose faces are congruent regular polygons and whose polyhedral angles are equal.*

EXERCISES

1. Find the locus of a point equidistant from
   - **a.** two intersecting planes.
   - **b.** two parallel lines.
   - **c.** the sides of an angle. (Page 38, Exercise 25)
   - **d.** two intersecting lines.

2. In plane geometry, you have located certain points as the intersection of two loci. In that connection you discussed the problem, telling under what conditions no solution, one solution, or more than one solution is possible. In solid geometry, the intersection of two loci may itself be a locus consisting of one or more lines. Having found this locus, discuss the possible solutions as you did in plane geometry.
   - **a.** In a given plane find the locus of a point at a given distance from another given plane.
   - **b.** In a given plane find the locus of a point equidistant from two intersecting lines.
   - **c.** Find the locus of a point equidistant from two parallel planes and a given distance from a third plane.
   - **d.** Find the locus of a point equidistant from two given points and also equidistant from two parallel lines.
   - **e.** Find the locus of a point equidistant from the edges of a trihedral angle.

3. A point is 12 inches from each face of a right dihedral angle. How far is the point from the edge of the dihedral angle? What is the locus of the point?

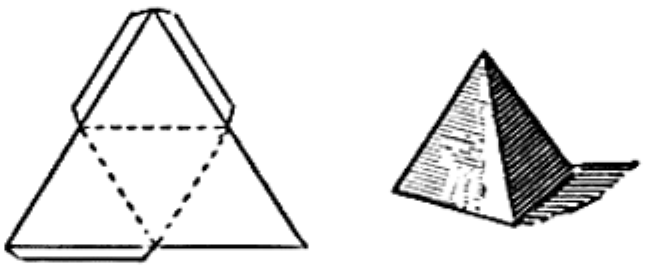
4. A point in the bisector of a 60° dihedral angle is 36 inches from the edge of the dihedral angle. How far is the point from the faces of the dihedral angle?

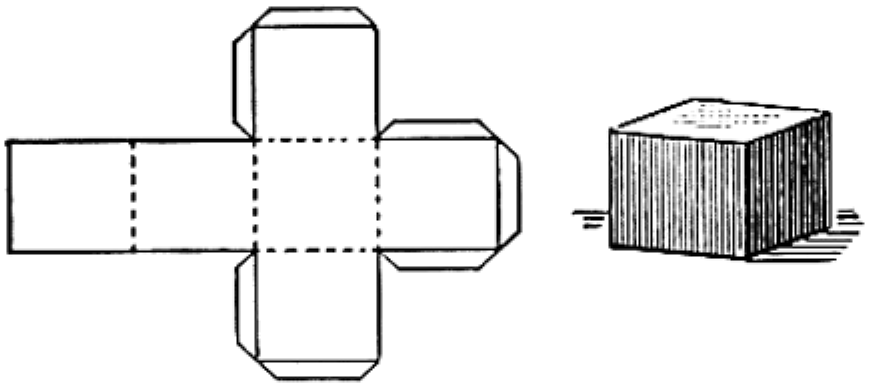
5. Prove that if a plane intersects four edges of a tetrahedron, bisecting three of them, then it bisects the fourth.

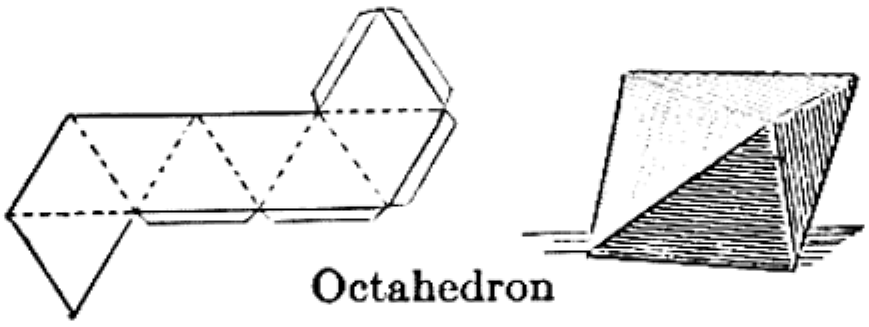
6. Considering the result you found in Exercise 8(c), page 65, show that the dihedral angle of a regular tetrahedron is greater than 60°.

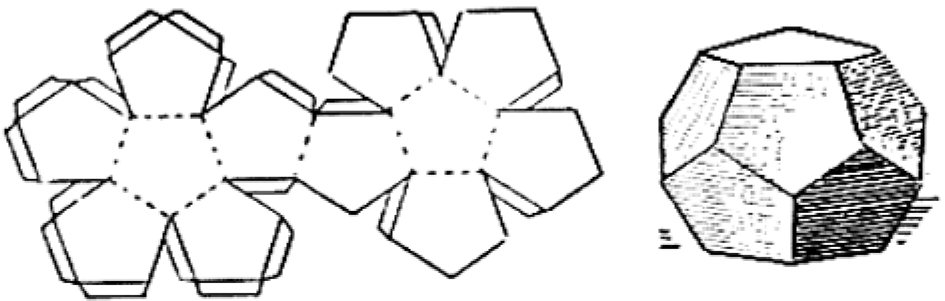
111. In Exercises 6 and 7 on page 64 you decided what polyhedral angles can be formed using the angles of regular polygons as the face angles. Since a regular polyhedron is bounded by congruent regular polygons, the existence of certain regular polyhedrons is partly determined by these possible polyhedral angles. For example, if no polyhedral angle can be formed with the angles of a regular hexagon as its face angles, then no regular polyhedron with hexagonal faces is possible.

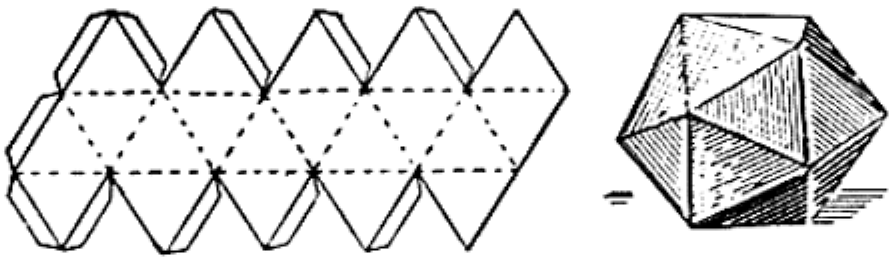
Considering, therefore, only the polyhedral angles, it appears that there are no more than five kinds of regular polyhedrons. Three of these would be bounded by equilateral triangles, one by squares, and one by regular pentagons. It can be proved deductively that these five regular polyhedrons do exist. You can prove it experimentally by actually constructing them from the patterns shown.


Tetrahedron


Hexahedron


Octahedron


Dodecahedron


Icosahedron

7. Copy several of the patterns in § 111 on heavy drawing paper, using a large scale. Cut out your patterns, fold them along the dotted lines, and paste them together. You will then have constructed the regular polyhedrons shown alongside the patterns.

8. Draw polyhedrons bounded by the following polygons (either regular or not):

a. 4 triangles.
b. 4 triangles and 1 quadrilateral.
c. 2 triangles and 3 quadrilaterals.
d. 5 triangles and 1 pentagon.
e. 6 quadrilaterals.
f. 5 quadrilaterals and 2 pentagons.

Copy the accompanying table.

In your Figure a, count the number of edges, vertices, and faces. Enter these numbers in the first row of *your* table in the columns *E* (number of edges), *V* (number of vertices), and *F* (number of faces).

Do the same for your other figures.

	E	*V*	*F*
Fig. a			
Fig. b			
Fig. c			
Fig. d			
Fig. e			
Fig. f			

The following formula is known as *Euler's Formula on Polyhedrons:*

$$E + 2 = V + F$$

Do the numbers in your table satisfy this formula? State this formula as a theorem.

Count the edges, the vertices, and the faces of the regular polyhedrons that you constructed in Exercise 7. Do these numbers satisfy Euler's formula?

SUPERIOR WORK

Mathematical Induction

S 112. *Mathematical induction* is a method sometimes used to prove that an algebraic formula is true for any number n. The method consists of the following two steps:

A. Assume that the formula is true for a general number and show that it must then be true for the next higher number.

B. Show that the formula is true for a particular number like 2. It must therefore be true for 3, for 4, for 5, . . .

Illustrative Example

Prove that $1 + 2 + 3 + 4 + \ldots + n = \dfrac{n(n+1)}{2}$

A. *Assume* that the formula is true for some number m, so that

$$1 + 2 + 3 + 4 + \ldots + m = \frac{m(m+1)}{2} \qquad (1)$$

Adding $m + 1$ to each side of the equation, we have

$$1 + 2 + 3 + 4 + \ldots + m + (m + 1) = \frac{m(m+1)}{2} + (m + 1)$$

Simplifying the right-hand member,

$$1 + 2 + 3 + 4 + \ldots + (m + 1) = \frac{(m+1)(m+2)}{2} \qquad (2)$$

This shows that if the formula is true for m, it is true for $m + 1$.

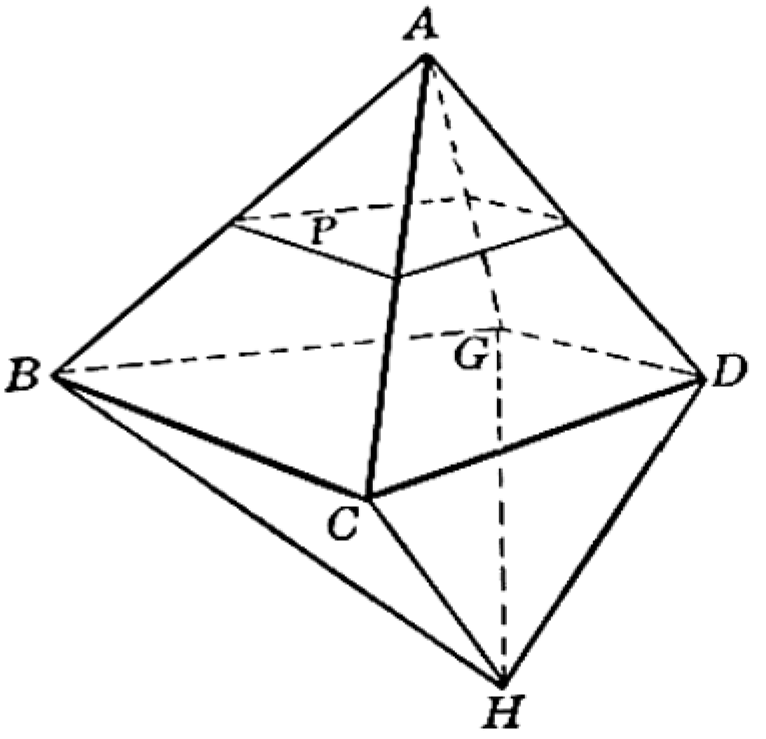
B. By direct substitution, the formula is obviously true when $n = 2$. Therefore it is true for 3, then for 4, then for 5, etc.

Illustrative Example

Using mathematical induction, prove Euler's formula (Exercise 8).

A. *Assume* that Euler's formula is true for a general polyhedron, such as $ABCDGH$, having E edges, V vertices, and F faces; that is,

$$E + 2 = V + F \qquad (1)$$



Let us add one more face to the polyhedron by cutting away the corner A with the plane P. In the new polyhedron, let F', V', and E' be the number of faces, vertices, and edges respectively.

Then $F' = F + 1$ or $F = F' - 1$

Assuming that n faces originally met at A, then P cuts these faces in n lines. Thus, n edges have been added to the polyhedron.

Then $E' = E + n$ or $E = E' - n$

The new polyhedron has gained n vertices where P cuts the n edges, but it has lost vertex A. The net gain is $(n - 1)$ vertices.

Hence $$V' = V + (n - 1) \quad \text{or} \quad V = V' - n + 1$$

Substituting these new values for E, V, and F in (1), we have

$$(E' - n) + 2 = (V' - n + 1) + (F' - 1)$$

Simplifying, $$E' + 2 = V' + F' \tag{2}$$

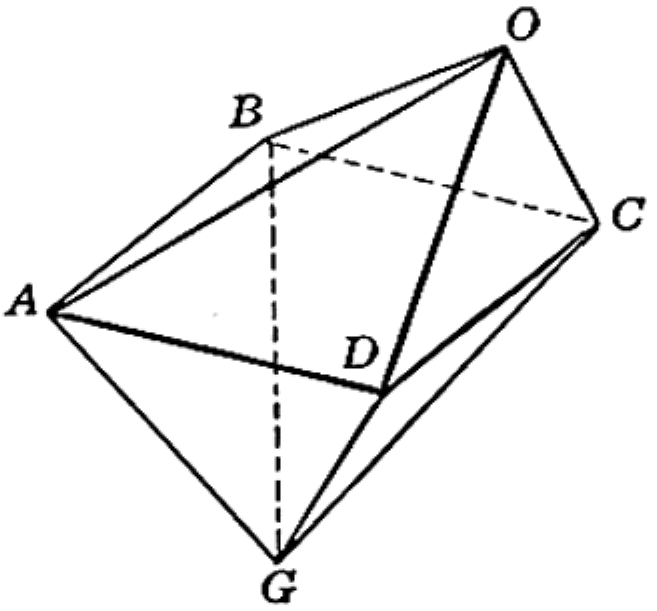
This shows that if the formula holds for a polyhedron with F faces, it holds for a polyhedron with one more face.

B. Since the formula is true for a tetrahedron where $F = 4$, it must be true for $F = 5, 6, 7$, etc.

S 1. If V is the number of vertices of a polyhedron and S is the sum of the face angles, prove by mathematical induction that

$$S = (V - 2)\ 2 \text{ st. } \measuredangle\text{s}.$$

Help: Assume that the formula is true for a polyhedron $ABCDG$, having V vertices. Increase by 1 the number of vertices by adding a vertex O (and the necessary number of triangles). Show that the formula holds true for this new polyhedron.



S 2. Prove Euler's formula by using the construction in S 1.

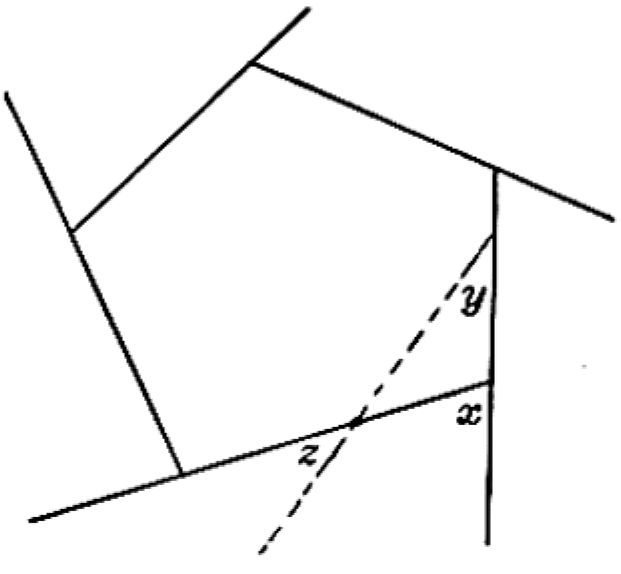
S 3. Prove the following formulas by mathematical induction:

a. $2 + 4 + 6 + \ldots + 2n = n(n + 1)$
b. $2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$

S 4. Prove by mathematical induction that the sum of the exterior angles of a plane polygon is two straight angles.

Help: Assume that the proposition is true for a polygon of m sides. Increase the number of sides of the polygon by one, by drawing the dotted line in the figure.

Show that the sum of the exterior angles in the new polygon is the same, since the discarded exterior angle x equals the sum of the new exterior angles y and z.



S 5. Prove the following proposition about numbers by mathematical induction:

If the sum of the digits of a number is divisible by 9, the number itself is divisible by 9.

Help: Assume that the proposition is true for some number n. Now show that adding 9 to the number to get the next multiple of 9 will either increase the sum of the digits by 9 or leave it unchanged.

S 6. In § 2, the nature of inductive thinking was explained briefly. The following quotations from *Dynamic Plane Geometry* show the relationship between inductive and deductive thinking.

"One disadvantage of the inductive method of reasoning is that the resulting general proposition is not absolutely certain."

"Another drawback of the inductive method is that the method does not tell why a proposition is true."

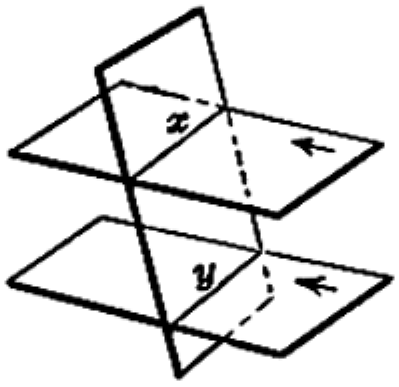
"The inductive method has the great advantage that it leads to discovery more often than the deductive method."

"The inductive method is based on the assumption that nothing happens without a cause. This assumption is sometimes called the *postulate of knowledge.* In inductive reasoning we conclude that, if a proposition is true a great many times, there must be an unknown reason that will make it true generally. The postulate of knowledge gives us the connecting link between the inductive and the deductive methods; because once the unknown reason becomes known, we have a deductive proof of our proposition."

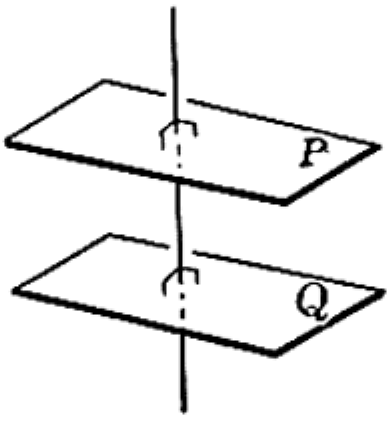
a. Compare mathematical induction with inductive thinking as to the certainty of the conclusion in each.
b. Does mathematical induction tell why a proposition is true?
c. Does mathematical induction lead to discovery?
d. Does mathematical induction depend on the postulate of knowledge?
e. Would you classify mathematical induction as inductive or deductive thinking? Why?

SUMMARY

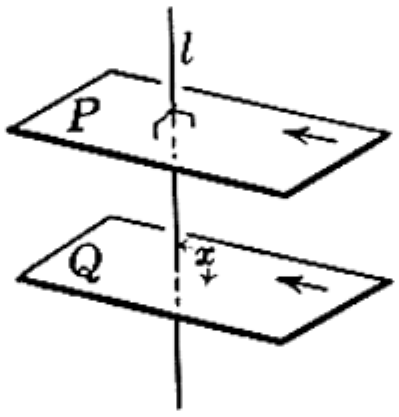
113. The diagrams below illustrate some important propositions about parallel planes.



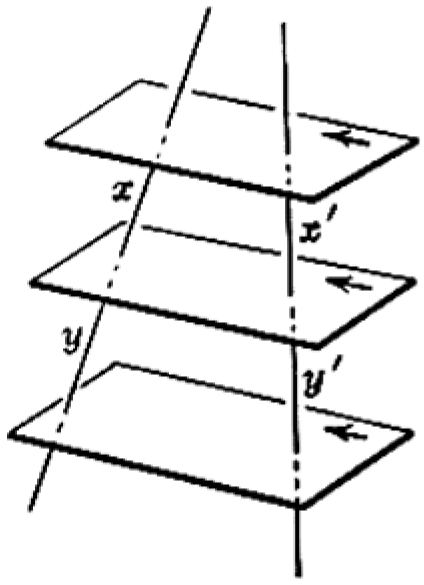
$x \parallel y$ (§ 54)



$P \parallel Q$ (§ 57)

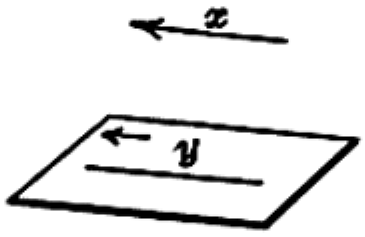


$l \perp Q$ (§ 58)

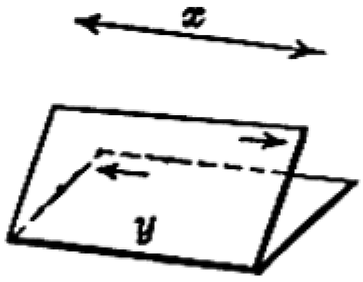


$\frac{x}{y} = \frac{x'}{y'}$ (§ 61)

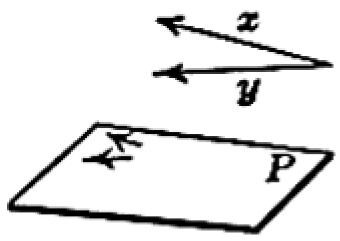
The following illustrate propositions about a line parallel to a plane.



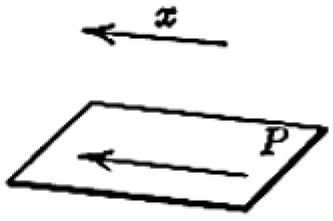
(x and y coplanar)
$x \parallel y$ (§ 64)



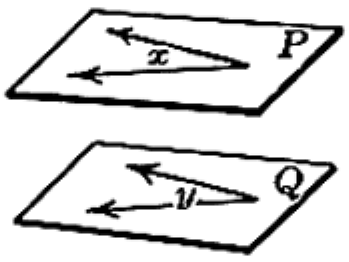
$x \parallel y$ (§ 65)



x and y determine
$Q \parallel P$ (§ 66)

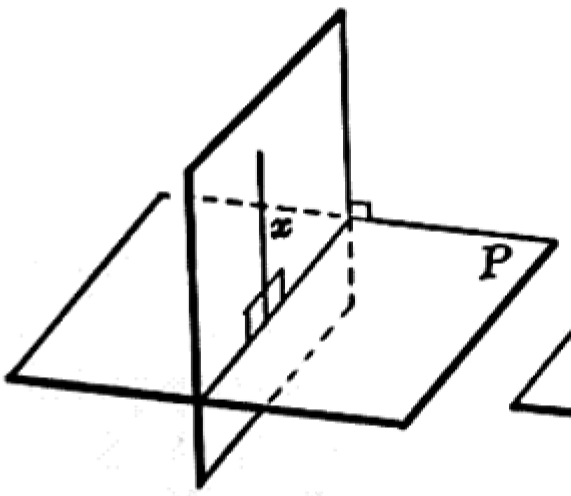


$x \parallel P$ (§ 67)

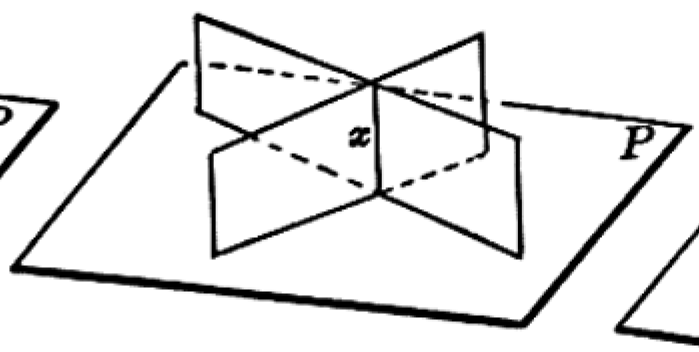


$P \parallel Q$
$\angle x = \angle y$ (§ 68)

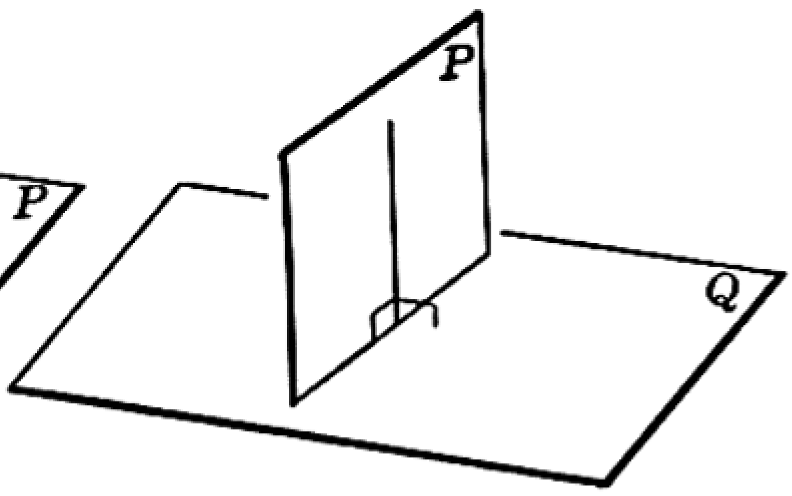
The following illustrate propositions about a line perpendicular to a plane.



$x \perp P$ (§ 82)



$x \perp P$ (§ 85)



$P \perp Q$ (§ 88)

The following classification of important theorems by conclusions will be found useful in proving exercises. P and Q denote planes· l and m lines.

1. $P \parallel Q$ if
   - **a.** P and Q are $\perp$ to the same line. (§ 57)
   - **b.** P is $\parallel$ to two intersecting lines in Q. (§ 66)
   - **c.** P contains two intersecting lines respectively $\parallel$ to two intersecting lines in Q. (§ 68)
2. $P \parallel l$ if
   - **a.** P contains a line $\parallel$ to l. (§ 67)
   - **b.** P and l are each $\perp$ to the same plane. (§ 84)
3. $l \parallel m$ if
   - **a.** l and m are $\perp$ to the same plane. (§ 48)
   - **b.** l and m are each $\parallel$ to the same line. (§ 50)
   - **c.** l and m are the intersections of two parallel planes with a third plane. (§ 54)
   - **d.** l is the intersection of a plane containing m with a plane $\parallel$ to m. (§ 64)
   - **e.** l is the intersection of two planes each $\parallel$ to m. (§ 65)
4. $P \perp Q$ if
   - **a.** P contains a line $\perp$ to Q. (§ 88)
   - **b.** P is parallel to a line $\perp$ to Q. (§ 90)
5. $l \perp P$ if
   - **a.** l is $\perp$ to two intersecting lines in P. (§ 30)
   - **b.** three points in P are each equidistant from two points in l. (§ 43)
   - **c.** l is $\parallel$ to another line which is $\perp$ to P. (§ 49)
   - **d.** l is $\perp$ to a plane which is $\parallel$ to P. (§ 58)
   - **e.** l is in another plane which is $\perp$ to P, and is $\perp$ to the intersection of the two planes. (§ 82)
   - **f.** l is the intersection of two planes each $\perp$ to P. (§ 85)

We also proved the following propositions in this chapter:

Through a given point one, and only one, plane may be constructed parallel to a given plane or parallel to two skew lines.

Through one of two skew lines one, and only one, plane may be constructed parallel to the other line.

A common perpendicular may be constructed to two skew lines.

Analogous to the sides of a triangle, the sum of two face angles of a trihedral angle is greater than the third face angle.

Unlike the angles about a point in a plane, the sum of the face angles of a polyhedral angle is less than 360°.

Analogous to the bisector of a plane angle, the bisector of a dihedral angle is the locus of a point equidistant from the faces.

Analogous to the circumcenter and the incenter of a triangle, there is a point equidistant from the vertices, and there is another point equidistant from the faces of a tetrahedron.

Unlike plane polygons, two solid figures like two polyhedral angles or two polyhedrons are congruent only if the equal parts are arranged in the same order; if the equal parts are in the opposite order, the solids are symmetric.

RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, go back to the section indicated.

- parallel planes § 53
- line parallel to plane § 63
- half plane § 71
- dihedral angle(s) § 72
  - edge § 71, faces § 73
  - plane angle, right section §§ 74–75
  - measure § 77
  - right, acute, obtuse § 79
  - adjacent, vertical, complementary, supplementary § 79
- perpendicular planes § 79
- projection of point § 87
- projection of line § 87
- projecting line § 87
- plane of projection § 87
- projecting plane § 93
- angle between line and plane § 95
- polyhedral angle(s) § 97
  - vertex, faces § 97
  - convex § 98
- trihedral, tetrahedral, pentahedral angle § 98
- polyhedron § 101
  - faces, vertices, diagonals, convex § 101
- tetrahedron § 101
- hexahedron § 101
- octahedron § 101
- dodecahedron § 101
- icosahedron § 101
- congruent solids § 109
- symmetric solids § 109

REVIEW EXERCISES

1. Complete the following statements.

a. Two lines, one in each of two parallel planes, cannot —?—.

b. Either the intersections of planes *P* and *Q* with a third plane are parallel or —?—.

c. For a line to be parallel to a plane which does not contain the line, it is —?— (necessary, sufficient) that it be parallel to any line in the plane.

d. For two planes to be parallel, it is —?— (necessary, sufficient) that their intersections with a third plane be parallel.

e. For two planes to be perpendicular, it is —?— (necessary, sufficient) that one plane contain a line perpendicular to the other.

2. Consider this proposition in plane geometry: If two lines *m* and *n* are each perpendicular to a line *t*, then *m* and *n* are parallel. Is this proposition valid if

   a. line *t* is changed to plane *t*?
   b. lines *m* and *n* are changed to planes *m* and *n*?
   c. lines *m* and *t* are changed to planes *m* and *t*?
   d. all the lines are changed to planes?

3. Consider this proposition in plane geometry: If a line *t* is perpendicular to one of two parallel lines *m* and *n*, then it is perpendicular to the other. Answer the same questions asked in Exercise 2.

4. A plane *P* contains a line *l*. What statement can you make about *P* if

   a. *l* is perpendicular to another plane *Q*?
   b. *l* is parallel to another plane *Q*?
   c. *l* is parallel to another line *m*?

5. Which of the following constructions are determined? If a construction is not determined, tell whether it is impossible or whether more than one construction is possible. Note possible exceptions.

   a. A line through a given point, parallel to each of two intersecting planes.
   b. A plane through a given line, parallel to a given plane.
   c. A plane through a given point, perpendicular to two intersecting planes.
   d. A plane through a given point, parallel to two parallel planes.
   e. A plane perpendicular to a given plane and perpendicular to a given line oblique to the plane.
   f. A plane parallel to each of two given skew lines.

6. Can each of the following combinations be the face angles of a trihedral angle?

   a. 110°, 60°, 60° **b.** 160°, 110°, 100° **c.** 150°, 70°, 60°

   In each of the following, two face angles of a trihedral angle are given. Between what limits is the third face angle?

   d. 90°, 80° **e.** 120°, 70° **f.** 150°, 150°

7. How many kinds of regular polyhedrons are there? How many kinds are bounded by triangles? by squares? by pentagons? by hexagons?

8. **a.** If two polyhedrons have the same number of faces, do they have the same number of vertices?
   b. If two polyhedrons have the same number of faces and vertices, do they have the same number of edges?
   c. Between what limits is the sum of the dihedral angles of a trihedral angle? of a tetrahedral angle? of a pentahedral angle?

9. Prove: If a plane bisects the legs of a trapezoid, it is parallel to the bases.

10. Prove: If a plane parallel to one side of a triangle passes through the intersection of two medians of the triangle, it divides the other two sides in the ratio of 2:1.

11. Prove: If a plane bisects two adjacent sides of a (plane or skew) quadrilateral, it is parallel to the line that bisects the other two sides.

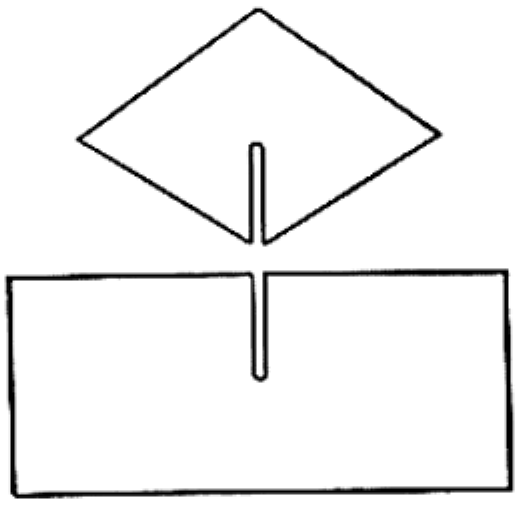
12. Prove: If a plane is perpendicular to the intersection of two planes, it is perpendicular to both planes.

13. Prove: If a line and a plane are both perpendicular to the same line, they are parallel.

14. Prove: If two perpendicular planes are cut by a plane parallel to their intersection, the acute dihedral angles are complementary.

15. Prove that the ratio of two collinear line segments from a fixed point to two fixed parallel planes is constant.

16. Prove: Three non-collinear points each equidistant from the faces of a dihedral angle determine the plane bisecting the dihedral angle.



17. a. Cut out a rhombus from a card. Cut a slot in it extending half the length of the smaller diagonal as shown in the figure. Fit this slot into a slot made in another card.

Using your model, show that the projection of a rhombus can be a square.

b. If the diagonals of the rhombus in your model are 4 inches and 3 inches, find the area and the perimeter of your projection when it is a square.

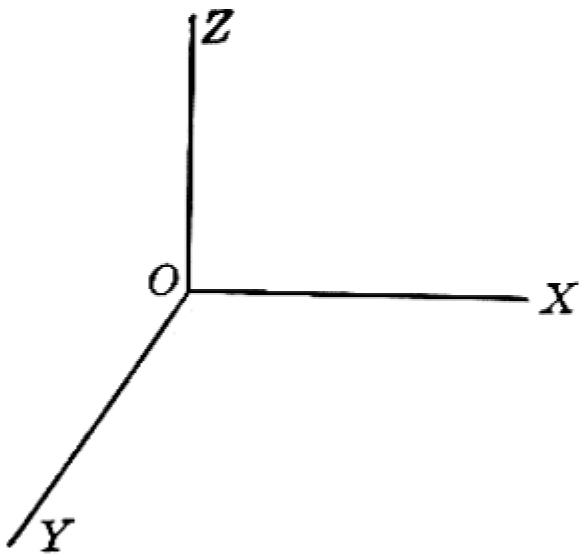
c. Can the projection of a square be a rhombus?

18. A rectangle 14 inches by 10 inches is projected on a plane. One of the 14-inch sides is in the plane of projection, and the 10-inch side makes an angle of 45° with that plane. Find the perimeter and the area of the projection.

19. A trapezoid with bases 28 inches and 18 inches, and legs 13 inches each, is projected on a plane. The bases of the trapezoid are parallel to the plane of projection, but the plane of the trapezoid makes an angle of 25° with the plane of projection. Using a table of cosines, find the area of the projection. (See page 59, Exercise 8.)

20. A point is projected on each of two intersecting planes. The projecting lines are each $7\frac{1}{2}$ inches long and form an angle of 120°. How far is the point from the intersection of the planes?

SUPERIOR WORK



S 1. GIVEN: Three intersecting lines in space OX, OY, and OZ.
$OX \perp$ plane OYZ.
$OY \perp$ plane OXZ.

PROVE: $OZ \perp$ plane OXY.

S 2. Prove: If three parallel or concurrent lines intersect two parallel planes, the line segments joining the points of intersection in one of the parallel planes are proportional to the corresponding line segments in the other plane. (Cf. page 45, S 2.)

S 3. Prove: If two dihedral angles have their faces respectively perpendicular and their edges parallel, it is possible to draw a circle intersecting the four lines in which the planes intersect. (See page 60, S 6.)

S 4. The angle between two lines in space is defined as the angle between two intersecting lines which are parallel respectively to the two given lines.

- **a.** Can the angle between two skew lines be 0°?
- **b.** What is the angle between two parallel lines?
- **c.** Compare two skew lines with two parallel lines with respect to the number of common perpendiculars and their lengths and with respect to the angles between the lines.

TEST

1. Prove that two parallel planes intercept equal arcs on a circle.
2. Prove: If a plane bisects two sides of a triangle, it is parallel to the third side.
3. Prove: The plane of the plane angle of a dihedral angle is perpendicular to the faces of the dihedral angle.
4. Prove: If two parallel planes intersect two other parallel planes, the opposite dihedral angles are equal.
5. The projections of a point on the faces of a 120° dihedral angle are each 10 inches from the edge of the dihedral angle. How long are the projecting lines?
6. Through a given point construct a line parallel to two intersecting planes.
7. Which of the following statements are true and which are false?
   - **a.** Through a given point there is one, and only one, line parallel to a given plane.
   - **b.** A skew quadrilateral cannot have four right angles.

c. If three planes meet, the intersections are either parallel or concurrent.

d. If two planes are perpendicular to a third plane, they are parallel.

e. If a line is perpendicular or parallel to each of two planes, the planes are parallel.

f. If a plane R is perpendicular to each of two intersecting planes P and Q, the intersections in R form an angle which is the measure of the angle between P and Q.

8. Complete each of the following statements:

a. If two lines perpendicular to two separate planes form an angle of 70°, the planes form angles of —?—.

b. If two face angles of a trihedral angle are 70° and 120°, the third face angle is greater than —?— degrees and less than —?— degrees.

c. Through a line perpendicular to a given plane, —?— plane(s) can be drawn perpendicular to the given plane.

d. The locus of a point equidistant from points A and B and from points C and D is usually a(n) —?— in plane geometry but a(n) —?— in solid geometry.

e. For three planes to cut proportional segments on two lines, it is —?— (necessary, sufficient) that the planes be parallel.

f. For the projections on a plane of three points to be in a straight line, it is —?— (necessary, sufficient) that the points be collinear.

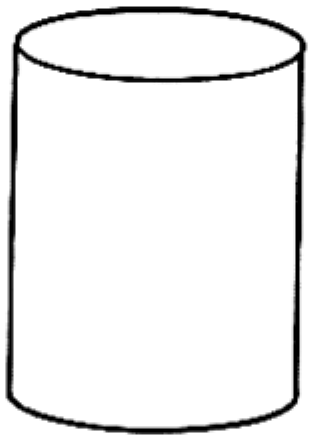
3 Translating a Line

Properties of Prisms and Cylinders

FORMATION OF PRISMS AND CYLINDERS

PREPARATORY EXERCISE

1. Lay your pencil on a flat sheet of paper.
   - **a.** Can you turn your pencil while it is in full contact with the paper?
   - **b.** Can you move the pencil so that it remains parallel to its original position and still have it remain on the paper?
   - **c.** Can a moving straight line contain every point of the flat surface of your paper? If so, you can say that the line generates the surface.
   - **d.** Can a straight line generate a plane by turning about a point and always intersecting another straight line? another curve? Explain your answers.
   - **e.** Can a straight line generate a plane by moving parallel to itself and always intersecting another straight line? another curve? Explain your answers.



2. Roll up your paper so that two opposite edges fit together as shown in the figure. Paste these edges together. Your paper is now the curved surface of a cylinder.
   - **a.** Can you keep your pencil in contact with any part of your curved surface?
   - **b.** Can you turn your pencil and still keep it in contact with the curved paper throughout the length of the pencil?
   - **c.** Can you move your pencil and still keep it in full contact with your curved paper? If so, how?
   - **d.** Can a moving straight line contain every point of your curved surface? If so, you can say that a straight line generates your surface.
   - **e.** Can a moving straight line generate the surface of a cylinder and always intersect another straight line? another curve?
3. A surface that can be generated by moving a straight line is called a *ruled* surface.
   - **a.** Name some ruled surfaces and describe the motion of the generating line.
   - **b.** Is a sphere a ruled surface?
   - **c.** Can you generate the complete surface of a chalk box by moving a straight line so that it always remains parallel to itself? If not, what part of the surface can you generate in this way?

d. Can you generate the complete surface of a closed can in the form of a cylinder by moving a straight line so that it always remains parallel to itself?

e. The various positions of a straight line that generates a ruled surface are called the *rulings* of the surface. Can the surface of a cylinder have more than one set of rulings? Can the surface of a plane have more than one set of rulings?

114. We saw that a moving straight line may generate a plane (§ 7). It may also generate other kinds of surfaces, known as *ruled surfaces.* The kind of surface that a moving straight line generates depends on two things: (1) the way the straight line moves, and (2) the kind of curved or broken line that it intersects while in motion.

In this chapter we shall study ruled surfaces that are generated when the moving line remains parallel to a fixed line. This kind of motion is called *translation.* The diagrams on the next page show that a translated line, called the *generatrix*, keeps touching a broken or curved line, called the *directrix.* This generates a *prismatic* or a *cylindrical* surface which is finally made into a *prism* or a *cylinder.* As you read the definitions, follow them with the diagrams.

Definitions

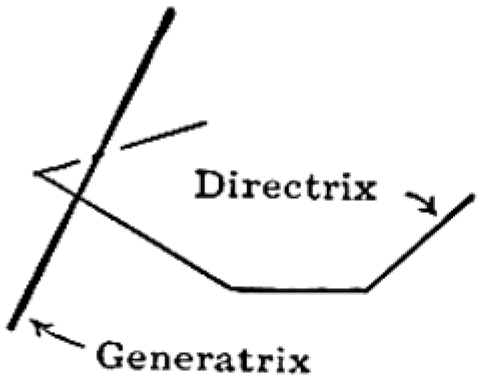
115. *To translate a line or a plane means to move it so that it remains parallel to its original position.*

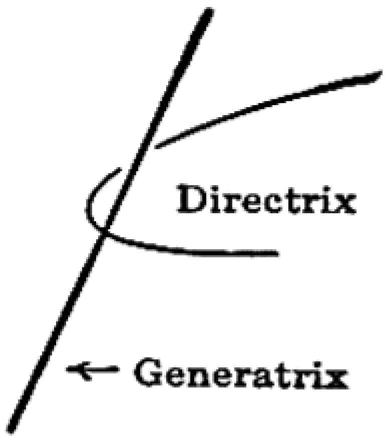
116. *A ruled surface is a surface that can be generated by moving a straight line. The moving straight line is called the generatrix. The various positions of the generatrix are called the elements (or the rulings) of the surface. If the generatrix always passes through a curved or broken line in another plane, that curved or broken line is called the directrix of the surface.*

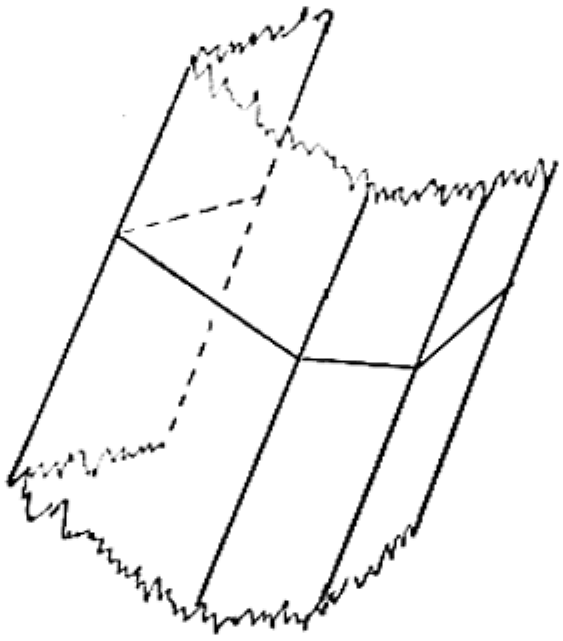
117. *A prismatic surface is a ruled surface in which the generatrix is translated and the directrix is a broken line. A cylindrical surface is a ruled surface in which the generatrix is translated and the directrix is a curved line.*

118. *A closed prismatic or cylindrical surface is one in which the directrix is closed.*

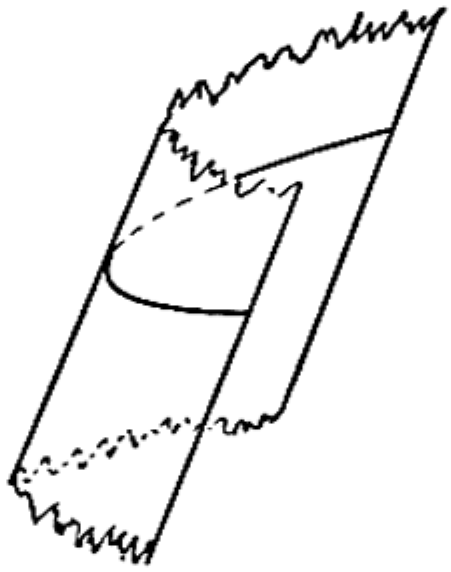
119. *A prism is a solid bounded by a closed prismatic surface and two parallel planes cutting all the elements. A cylinder is a solid bounded by a closed cylindrical surface and two parallel planes cutting all the elements.*



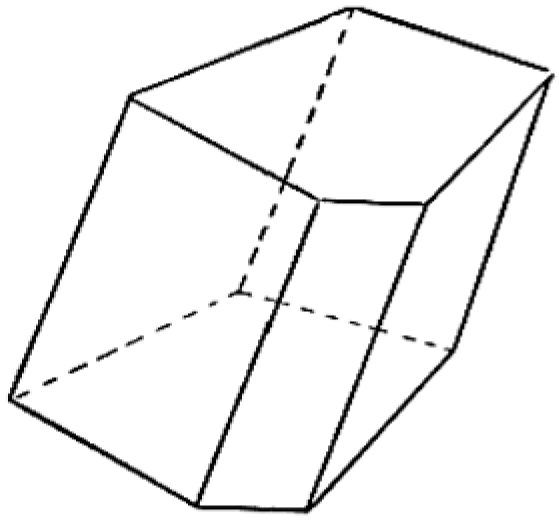




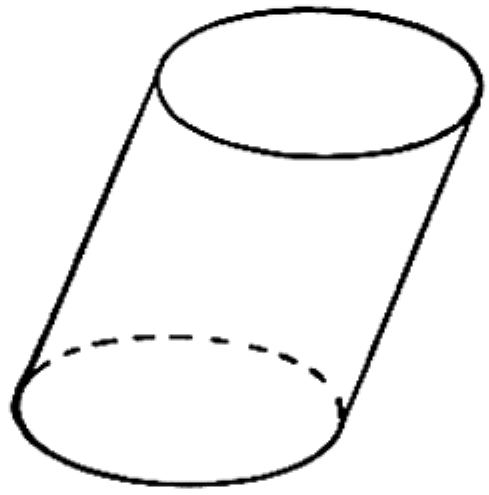
Prismatic Surface



Cylindrical Surface



Prism



Cylinder

120. *In a prism or a cylinder, the parts of the parallel planes included by the prismatic or cylindrical surface are called the bases; the part of the prismatic or cylindrical surface included between the bases is called the lateral surface; the parts of the elements of the prismatic or cylindrical surface included between the bases are called the elements of the prism or the cylinder.*

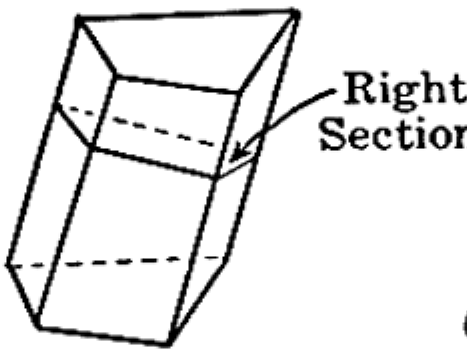
121. *The faces of a prism other than the bases are called the lateral faces.*

122. *An element of a prism that passes through a vertex of a base is called a lateral edge.*

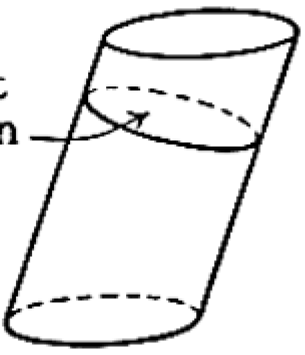
123. *An altitude of a prism or a cylinder is the perpendicular distance between the bases.*

124. *A right section of a prism or a cylinder is a section made by a plane which cuts all the elements and is perpendicular to one of them.*

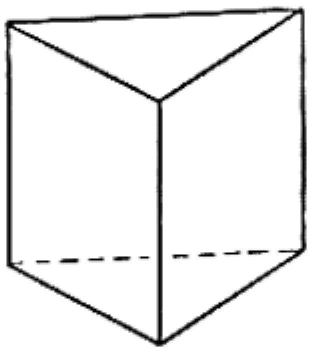
125. *A right prism or a right cylinder is one in which the bases are right sections. Otherwise the prism or the cylinder is oblique.*



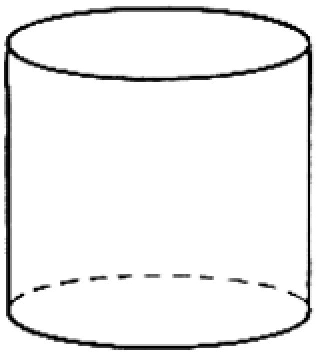
Oblique Prism



Oblique Cylinder



Right Prism



Right Cylinder

126. *A prism is triangular, quadrangular, pentagonal, etc., according as its bases are triangles, quadrilaterals, pentagons, etc.*

127. *A circular cylinder is a cylinder in which the bases are circles. The line segment joining the centers of the bases is called the axis of the cylinder.*

Unless otherwise stated, we shall refer to a circular cylinder simply as a cylinder.

EXERCISES

1. May a plane be considered a special case of a ruled surface? If so, what is the directrix and how does the generatrix move?
2. a. Why is a plane section of a prism a polygon?
   b. What is the least number of faces that a prism may have?
   c. A prism is a special case of what kind of solid?
3. Prove: If a plane is parallel to one element of a prism or a cylinder, it is parallel to all the elements except those that it may contain.
4. Prove that all the elements of a prism or a cylinder are equal.
5. Prove that the corresponding sides of the bases of a prism are parallel.
6. What kind of polygons are the lateral faces of a prism? Prove your answer.
7. a. Prove that parallel sections of a prism cutting all the lateral edges are congruent.
   b. Prove that the bases of a prism are congruent.
8. Prove that corresponding diagonals of the bases of a prism are parallel and equal.
9. a. Why is a right section of a prism or a cylinder perpendicular to all the elements?
   b. Prove that the lateral faces of a right prism are perpendicular to the bases.

10. What kind of prism or cylinder has the altitude equal to an element?

11. a. Prove: If two adjacent lateral faces of a prism are rectangles, the prism is a right prism.
    b. Prove: If two nonparallel lateral faces of a prism are rectangles, the prism is a right prism.

12. a. If the bases of an oblique prism are squares, what kind of polygon is a right section?
    b. Prove: If the bases of a prism are trapezoids, any section made by a plane that cuts all the lateral edges is a trapezoid.

13. In what kind of prism are all the lateral faces congruent?

14. Prove that the lateral edges of a prism make equal angles with the plane of either base.

15. Prove that the sum of the dihedral angles made by the lateral faces of a quadrangular prism equals 360°.

16. a. Draw a pattern, cut it out, fold and paste it to form a right triangular prism.
    b. Check Euler's formula $E + 2 = V + F$ (page 71, Exercise 8) in your model.
    c. Check the sum of the face angles in your model with the formula $S = (V - 2)2$ st. ∡ (page 73, S 1).

17. Prove that if a rectangle is rotated about one side as an axis, it generates a right circular cylinder.

18. Prove that the axis of a cylinder is parallel to the elements.

PROPERTIES OF THE PRISM AND THE CYLINDER

In the preceding exercises you have proved a number of important properties of prisms and cylinders. Some of these are now stated as corollaries.

Corollaries

128. All the elements of a prism or a cylinder are equal and parallel.

129. The lateral faces of a prism are parallelograms.

130. The sections of a prism formed by parallel planes cutting all the lateral edges are congruent.

131. The bases of a prism are congruent.

132. All the elements of a prism or a cylinder are perpendicular to the plane of a right section.

133. The lateral edges of a prism are perpendicular to the sides of a right section.

134. All the elements of a right prism or a right cylinder are perpendicular to both bases.

135. The lateral faces of a right prism are rectangles.

136. The altitude of a right prism or a right cylinder equals an element.

137. A right cylinder may be generated by revolving a rectangle about one side as an axis.

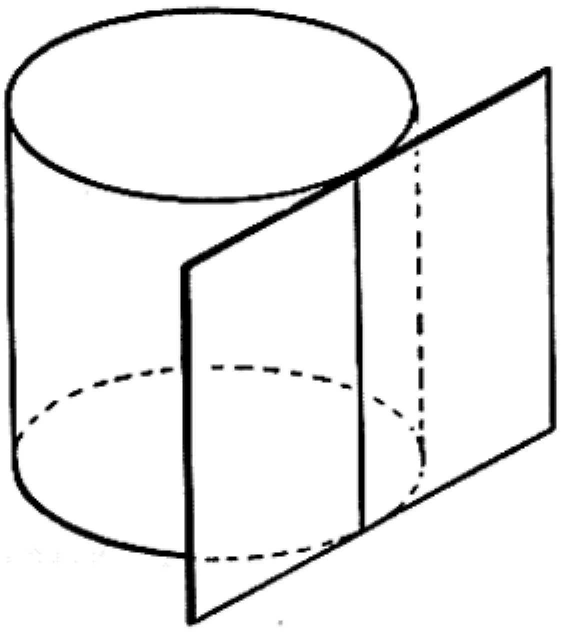
138. Definition. *A cylinder of revolution is a right circular cylinder.*

In plane geometry, we derived many properties of the circle from the properties of inscribed or circumscribed polygons. In solid geometry, we shall associate the cylinder with inscribed and circumscribed prisms. We therefore define how these prisms are related to the cylinder.

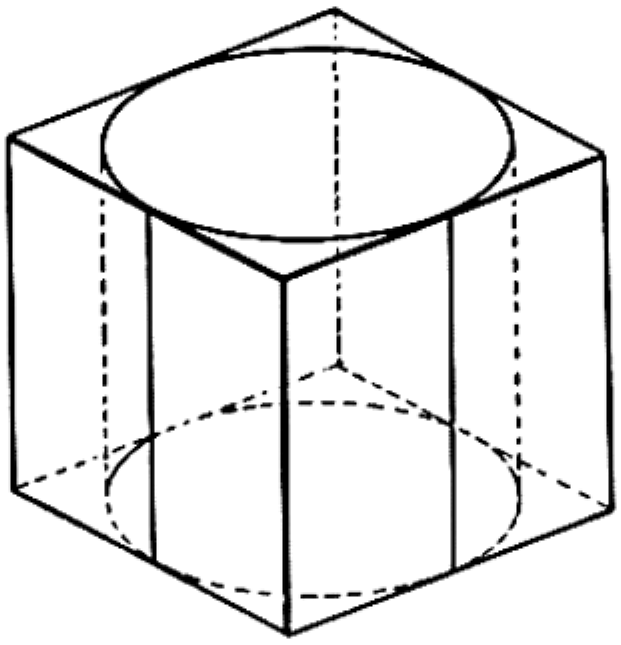
Point out how the next three definitions are analogous to the definitions § 415, § 420, and § 419 in plane geometry.

Definitions

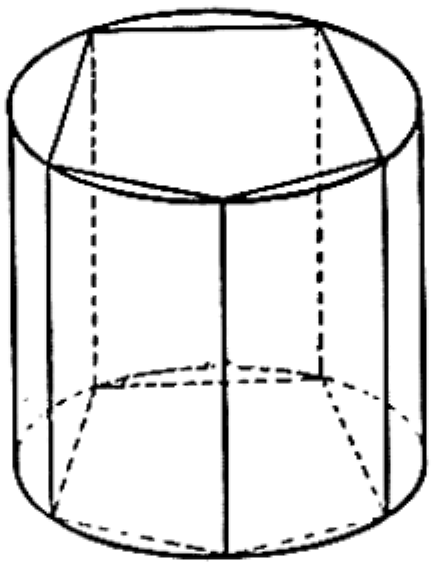
139. *A plane is tangent to a cylinder if it contains only one element of the cylinder.*



Plane Tangent to Cylinder



Prism Circumscribed about Cylinder



Prism Inscribed in Cylinder

140. *A prism is circumscribed about a cylinder if the bases of the prism are coplanar with the bases of the cylinder and if the lateral faces are tangent to the cylinder.*

141. *A prism is inscribed in a cylinder if the bases of the prism are coplanar with the bases of the cylinder and if the lateral edges of the prism are elements of the cylinder.*

142. *A regular prism is a right prism whose bases are regular polygons.*

From plane geometry we know that a regular polygon can be inscribed in or circumscribed about a circle. Hence we have the following corollary.

143. Corollary. A prism with regular bases can be inscribed in or circumscribed about a given cylinder.

144. In plane geometry we considered the properties of a regular inscribed polygon as the number of sides was steadily increased so that it approached the circle as a limit. We shall follow the same procedure with a prism of regular bases inscribed in a cylinder. Accordingly, we shall restate from plane geometry the idea of a variable approaching a limit and shall assume that certain properties of the variable prism are also the properties of the constant cylinder.

145. Definition. *A variable is said to approach a constant as a limit if it can be made to differ from the constant by a quantity as small as you please except zero.*

146. Corollary. If the number of lateral faces of an inscribed prism with regular bases is repeatedly doubled, the limit of the prism is the cylinder in which it is inscribed.

147. Postulate. *Any proposition about a prism with regular bases that does not depend on the number of lateral faces is equally valid for the circumscribed cylinder.*

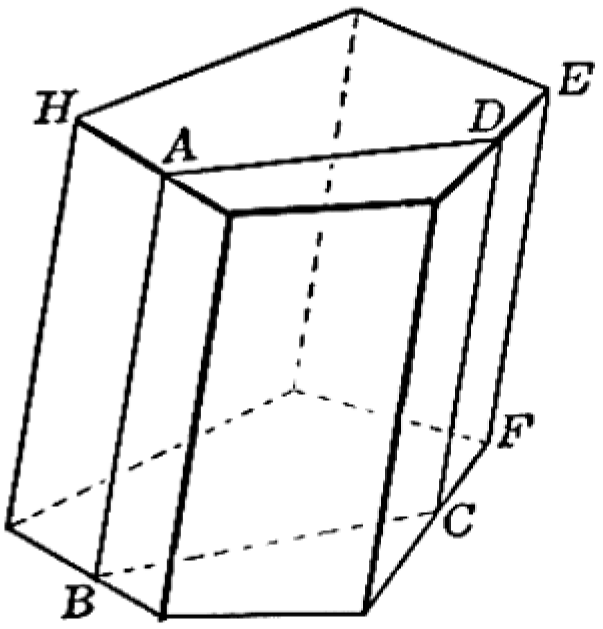
Applying this postulate to the properties of a prism stated in § 130 and § 131, we have the following properties of a cylinder.

148. Corollary. The sections of a cylinder formed by parallel planes cutting all the elements are congruent.

149. Corollary. The bases of a cylinder are congruent.

So far our main concern has been with the plane sections of a prism that cut the elements. The following theorem has to do with a plane section of a prism that contains an element.

150. THEOREM. A plane section of a prism which contains one element is a parallelogram.



GIVEN: Section $ABCD$ of prism HF formed by a plane containing element AB.

PROVE: $ABCD$ is a ▱.

PROOF:

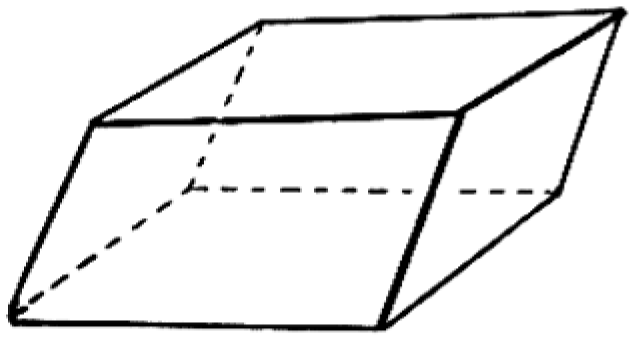
Statements	*Reasons*
1. $AB \parallel EF$.	1. Def. prismatic surface. (§ 117)
2. $AB \parallel$ plane DF.	2. A plane containing one of two ∥ lines is ∥ to the other. (§ 67)
3. $AB \parallel DC$.	3. If a line is ∥ to a plane, it is ∥ to the intersection of the plane with any plane containing the line. (§ 64)
4. $AD \parallel BC$.	4. The intersections of two ∥ planes with a third plane are ∥. (§ 54)
5. $ABCD$ is a ▱.	5. Def. ▱.

Since this theorem is true about a prism with regular bases and does not depend on the number of lateral faces, we may apply it to a cylinder and get the following proposition.

151. Corollary. A plane section of a cylinder which contains an element is a parallelogram.

152. Corollary. A section of a right prism or a right cylinder made by a plane passing through an element is a rectangle.

In some of the exercises that follow, you will derive certain properties of an important kind of prism called a *parallelepiped*.



153. Definition. *A parallelepiped is a prism whose bases are parallelograms.*

EXERCISES

1. How many degrees are in each dihedral angle formed by the lateral faces of a regular triangular prism? regular quadrangular prism? regular prism of n lateral faces?

2. a. Is it possible to have a plane touch only one point of a cylindrical surface? Why?
   b. If a prism is inscribed in or circumscribed about a cylinder, are the bases of the prism inscribed in or circumscribed about the bases of the cylinder? Why?
   c. What is the difference between a regular prism and a prism with regular bases?
   d. What kind of cylinder can be inscribed in or circumscribed about a regular prism?

3. a. How is the axis of a right cylinder related to the altitude of the cylinder?
   b. How would you define the axis of a prism with regular bases?

4. Prove: If a plane intersects a prism or a cylinder in one element, it intersects it again in another element.

5. a. What kind of figure is a plane section of a prism cutting only some lateral edges?
   b. What kind of figures are the plane sections of a circular cylinder cutting all the elements?
   c. What kind of figure is a plane section of a cylinder cutting only some elements?

6. a. Does an inscribed prism with regular bases become the same as the circumscribed cylinder after the number of lateral faces has been doubled a finite number of times? Why?
   b. Can the prism in (a) approach the circumscribed cylinder if one lateral face is kept constant while the remaining number of lateral faces is being doubled? Why?

7. a. How many and what kind of faces does a parallelepiped have?
   b. How many diagonals does a parallelepiped have? Are they necessarily equal?

8. a. Draw and cut out four parallelograms (not all congruent) which can be assembled to form the lateral faces of a parallelepiped. Paste them together with stickers. Now draw and cut out the necessary parallelograms for the bases of the parallelepiped and paste them on.
   b. Can any two opposite faces of your model be the bases of the prism? Demonstrate.
   c. Prove that the opposite faces of a parallelepiped are parallel and congruent. (Use § 131.)

9. a. If a parallelepiped is a right prism, it is called a *right parallelepiped*. How many faces of a right parallelepiped are necessarily rectangles?

b. If the bases of a right parallelepiped are rectangles, the figure is called a *rectangular solid* or a rectangular parallelepiped. How many faces of a rectangular solid are rectangles?

c. Is a rectangular prism the same as a rectangular parallelepiped?

10. Prove that a solid bounded by six rectangles is a rectangular solid as defined in Exercise 9.

11. Prove: The diagonals of a parallelepiped are concurrent and bisect each other.

12. The point where the diagonals of a parallelepiped meet is called the center of the parallelepiped. Prove that any line segment through the center of a parallelepiped terminating in two faces is bisected by the center.

SUPERIOR WORK

S 1. Prove: The lines connecting the vertices of one base of a triangular prism to the midpoints of the opposite edges of the other base meet in a point.

S 2. Construct a plane that will intersect an oblique circular cylinder in a rectangle.

S 3. A plane cuts the faces of a regular triangular prism so that its intersection with one lateral face is parallel to the base. Prove that the intersections of the plane with the other two faces are equal.

S 4. If a prism with regular bases is circumscribed about a cylinder, and if the number of faces is repeatedly doubled, does the size of the prism increase or decrease?

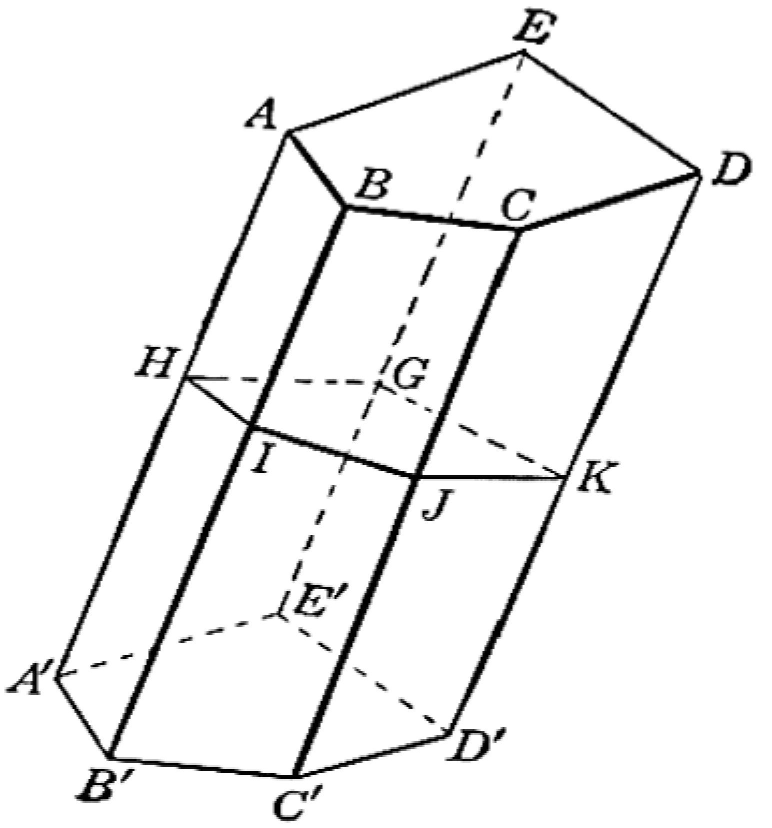
Comparing this variable circumscribed prism with the variable inscribed prism (in the same cylinder) also obtained by continually doubling the number of faces, what happens to the difference between them? Do they approach the same limit? What is the common limit?

Measuring Prisms and Cylinders

AREAS

We shall now learn how to find the area of the lateral surface, or the lateral area, of a prism. This surface consists of a number of parallelograms. Considering the lateral edges of the prism as the bases of these parallelograms, we shall find their respective altitudes to be the edges of a right section of the prism.

154. THEOREM. The lateral area of a prism equals the product of a lateral edge and the perimeter of a right section.



GIVEN:

L = lateral area of prism AD'.
e = one lateral edge of prism AD'.
p = perimeter of rt. section HK.

PROVE: $L = ep$.

PROOF: *Statements*	*Reasons*
1. $AA' = BB' = CC' = \cdots = e$.	1. All lateral edges of prism are =. (§ 128)
2. Quad. AB' is a ▱.	2. Lateral faces of prism are ▱s. (§ 129)
3. $HI \perp AA'$.	3. Lateral edges of prism are ⊥ to sides of rt. section. (§ 133)
4. ▱$AB' = e \cdot HI$. Similarly, ▱$BC' = e \cdot IJ$. ▱$CD' = e \cdot JK$. . . .	4. Area of ▱ = base × alt. (§ 509)
5. $L = e(HI + IJ + JK + \cdots)$.	5. Add. Ax.
6. $L = ep$.	6. Subs. Ax.

Since this theorem does not depend on the number of lateral faces of the prism, we may apply Postulate § 147 to it and get the corresponding proposition for a cylinder.

155. Corollary. The lateral area of a circular cylinder equals the product of an element and the perimeter of a right section.

156. Corollary. The lateral area of a right prism or a cylinder of revolution equals the product of its altitude and the perimeter of its base.

For a right cylinder, $L = hC$, where h is the altitude and C is the circumference of the base.

Or $L = 2\pi rh$, where r is the radius of the base.

157. Definition. *The total area of a prism or a cylinder is the sum of its lateral area and its bases.*

If T is the total area of a right cylinder,

$$T = 2\pi r^2 + 2\pi rh = 2\pi r(r + h)$$

The Parallelepiped

The parallelepiped is a fundamental figure in the measurement of solids. In the preceding set of exercises you have proved a number of properties of the parallelepiped. Some of these will now be stated as corollaries.

158. Corollary. The opposite faces of a parallelepiped are congruent and parallel.

Definitions

159. *A right parallelepiped is a parallelepiped that is a right prism.*

160. *A rectangular solid or a rectangular parallelepiped is a right parallelepiped whose bases are rectangles.*

161. *The lengths of three edges that meet at the same vertex of a rectangular solid are called its dimensions, or its length, width, and height.*

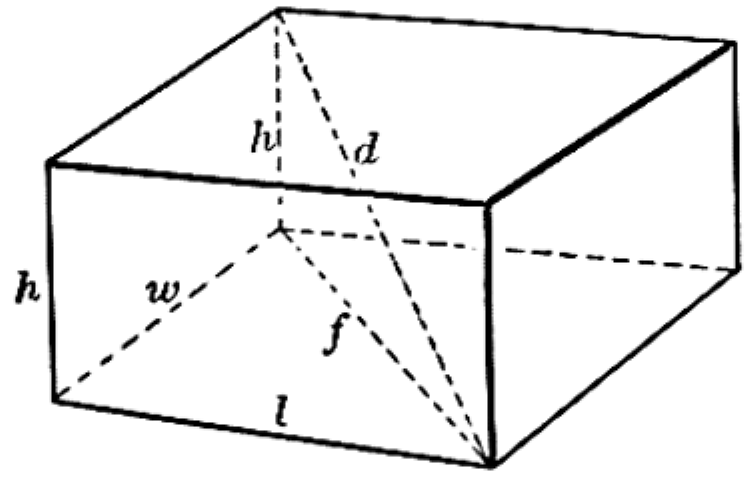
162. *A cube is a rectangular solid whose dimensions are equal.*

163. Corollary. All the edges of a cube are equal.

164. Corollary. All the faces of a rectangular solid are rectangles, and all the faces of a cube are squares.

The following is an extension of the Pythagorean Theorem to three-dimensional space.

165. THEOREM. The square of a diagonal of a rectangular solid equals the sum of the squares of the three dimensions.



GIVEN: d is a diagonal of rectangular solid with dimensions l, w, and h.

PROVE: $d^2 = l^2 + w^2 + h^2$.

CONS.: Draw face diagonal f.

1. $d^2 = f^2 + h^2$. Why?
   $f^2 = l^2 + w^2$.
2. $d^2 = l^2 + w^2 + h^2$. Why?

166. Corollary. All the diagonals of a rectangular solid are equal.

167. Definition. *A truncated prism is a part of a prism included between a base and a section of a plane not parallel to the base and cutting all the lateral edges.*

Why are both bases of a truncated parallelepiped parallelograms?

EXERCISES

1. A right section of a prism is a regular pentagon whose side is 4 inches. If a lateral edge of the prism is 5 inches, find the lateral area.
2. The radius of a circular right section of a cylinder is 2 inches. Find the lateral area of the cylinder if an element is 6 inches.
3. The base of a right prism is a square whose side is $6\frac{1}{2}$ inches. Find the lateral area of the prism if its altitude is 12 inches.
4. Find the total area of a regular triangular prism if its altitude is 16 inches and a side of its base is 6 inches.
5. The radius of a right cylinder is 7 inches and its altitude is 12 inches. Find the lateral and total areas of the cylinder.
6. The diameter of a wire is .012 inch. How many square inches are in the surface of the wire which is 100 feet long?
7. Solve each of the formulas $L = 2\pi rh$ and $T = 2\pi r(r + h)$ for h in terms of the other letters.
8. The total area of a right cylinder is 600 square inches, and the radius of the base is 6 inches. Find the altitude of the cylinder.
9. The lateral area of a right cylinder is 500 square inches, and its altitude is 12 inches. Find the radius of its base.
10. The base of a right prism is a rhombus with diagonals 5 inches and 12 inches. Find the total area of the prism if its altitude is 20 inches.
11. The altitude of a regular hexagonal prism is 8 inches. If the radius of its base is 2 inches, find the total area of the prism.
12. A lawn roller is 3 feet long and 20 inches in diameter. How many revolutions will the roller make in covering 600 square feet?
13. A rectangle 5 inches by 3 inches is rotated about its longer side as an axis. Find the total area of the resulting cylinder.
14. **a.** A rectangle whose dimensions are a and b is rotated about a as an axis. Find the lateral and total areas of the resulting cylinder.
    b. If the rectangle is rotated about b as an axis, find the lateral and total areas.
    c. Find the ratio of the lateral areas and the ratio of the total areas of the cylinders in (a) and in (b).

15. The plane axis section of a right cylinder is a square of side d. Write a formula for the lateral area and a formula for the total area of the cylinder.

16. A regular quadrangular prism is circumscribed about a cylinder whose altitude is 14 inches and whose base is a circle of 4-inch radius. Find the lateral and the total area of the prism.

17. A regular hexagonal prism is inscribed in a cylinder. The altitude of the prism is 10 inches, and a side of its base is 4 inches. Find the lateral and the total area of the cylinder.

18. Find the total area of a rectangular solid if its dimensions are 9, 7, and 5 feet; if its dimensions are l, w, and h.

19. Find the diagonal of a rectangular solid if its dimensions are 3, 4, and 12 feet; if its dimensions are 30, 20, and 12 feet.

20. The diagonal of a rectangular solid is 17 feet, and two of its dimensions are 9 feet and 12 feet. Find the third dimension.

21. **a.** The edge of a cube is e. Find its diagonal d.
b. Prove that the legs of a 60°–30° right triangle can be an edge and a diagonal of a cube.

22. The diagonal of a cube is 9 inches. Find the total area of the cube.

23. What is the cost of zinc-lining the inside of an open tank 3 feet 6 inches long, 2 feet 4 inches wide, and 9 inches high at $2 a square foot?

24. The total area of a rectangular solid is 810 square inches. Find its dimensions if they are in the ratios 2:3:5.

25. The diagonal of a rectangular solid is 57 inches. Find its dimensions if they are in the ratios 15:10:6.

26. The three face angles of a trihedral angle of a parallelepiped are 60°, 90°, and 120°. The common edge of the first two angles is 6 inches, of the second and third angles 8 inches, and of the first and third angles 10 inches. Find the total area of the parallelepiped.

27. The total area of a rectangular solid is 52 square inches. Find the dimensions if they are consecutive integers.

28. **a.** A room is 30 feet long, 20 feet wide, and 12 feet high. A fly is to crawl the shortest distance between two far corners. How long is this distance?
b. If the fly in (a) can crawl 30 feet per minute on the floor but only 20 feet per minute on the wall, how long would it take the fly to crawl the shortest distance between two far corners?

29. **a.** What is the shape of each lateral face of a truncated prism?
b. Prove that, unless two of the lateral faces of a truncated prism are parallel, not more than one of them can be a parallelogram.

30. Prove: The perpendicular drawn to the right section of a truncated triangular prism from the intersection of the medians in the other section equals one-third the sum of the lateral edges.

VOLUMES

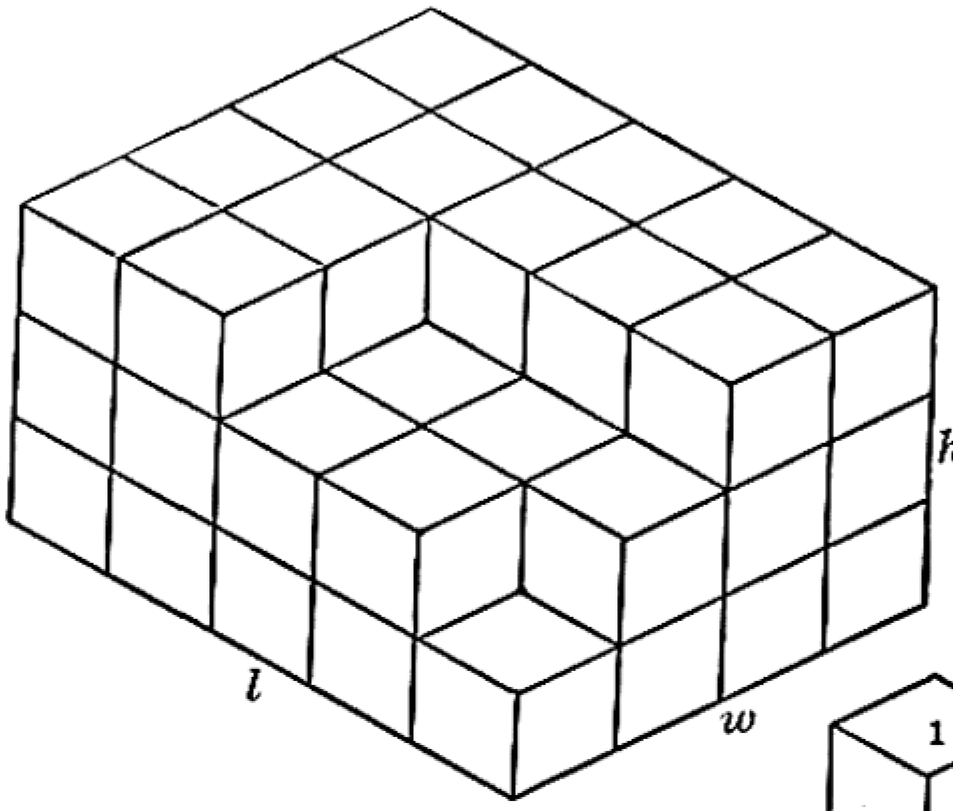
168. The size of the space enclosed by a solid figure is called its *volume.* To express the volume of a solid in numbers, we choose a *unit of volume.*

169. Definition. *A unit of volume is a cube whose edge is the unit of length.*

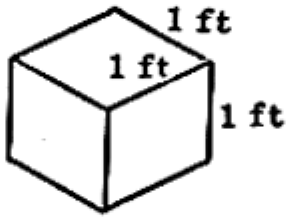
Thus, if the unit of length is a foot, the unit of volume is a cube whose edge is one foot. This unit of volume is called one cubic foot. Describe the terms cubic yard and cubic centimeter.

170. Definition. *The volume of a solid is the number of unit cubes that the solid contains.*

171. Definition. *Equal solids are solids that have equal volumes.*



$V=l\,w\,h$



1 Cubic Foot

From the figure it is evident that a rectangular solid whose dimensions are 5, 4, and 3 feet can be divided into unit cubes. There will be 3 horizontal layers of these cubes, each layer containing $5 \cdot 4$ cubes. The total number of these cubes will therefore be $5 \cdot 4 \cdot 3$, or the product of the three dimensions.

With any rectangular solid given, it may not be possible to measure all three dimensions in terms of a common unit of length. For example, two dimensions may be rational and the third dimension may be $\sqrt{6}$. Such dimensions are called *incommensurable.* We shall nevertheless assume that the volume of any rectangular solid whose dimensions are l, w, and h is equal to lwh.

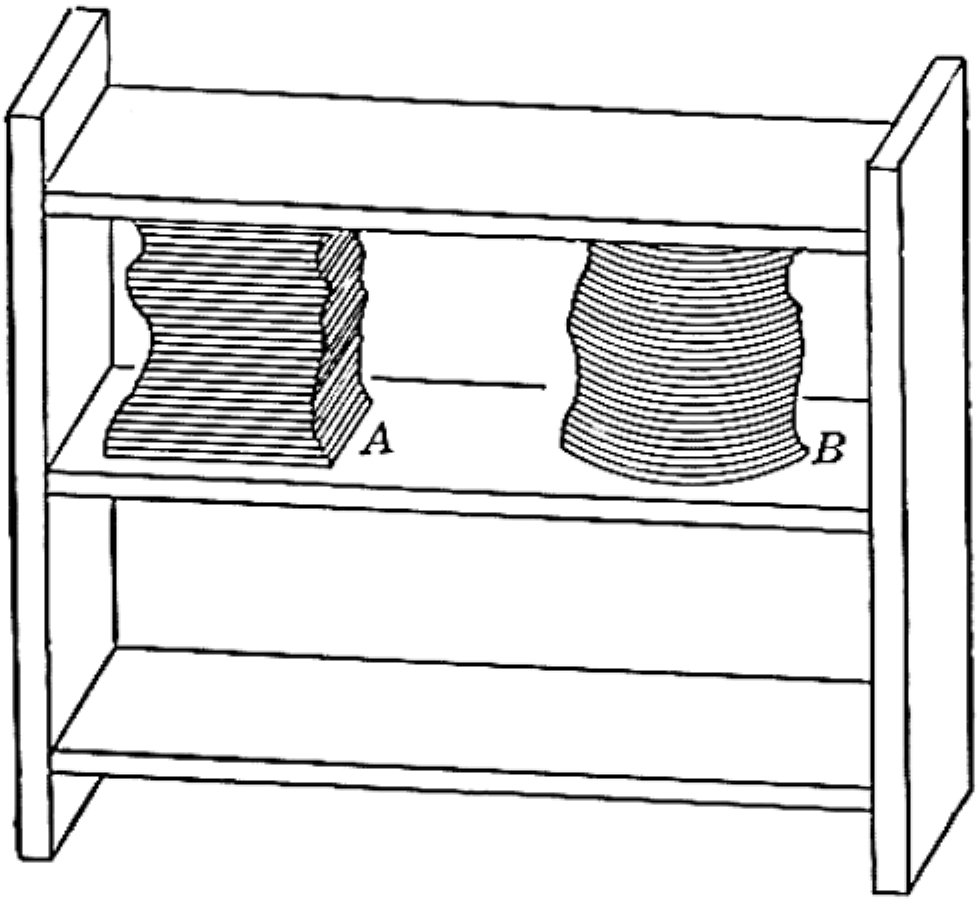
172. Postulate. *The volume of a rectangular solid equals the product of its three dimensions.*

173. Corollary. The volume of a rectangular solid equals the product of (the area of) the base and its altitude.

174. Corollary. The volume of a cube equals the cube of an edge.

Cavalieri Solids

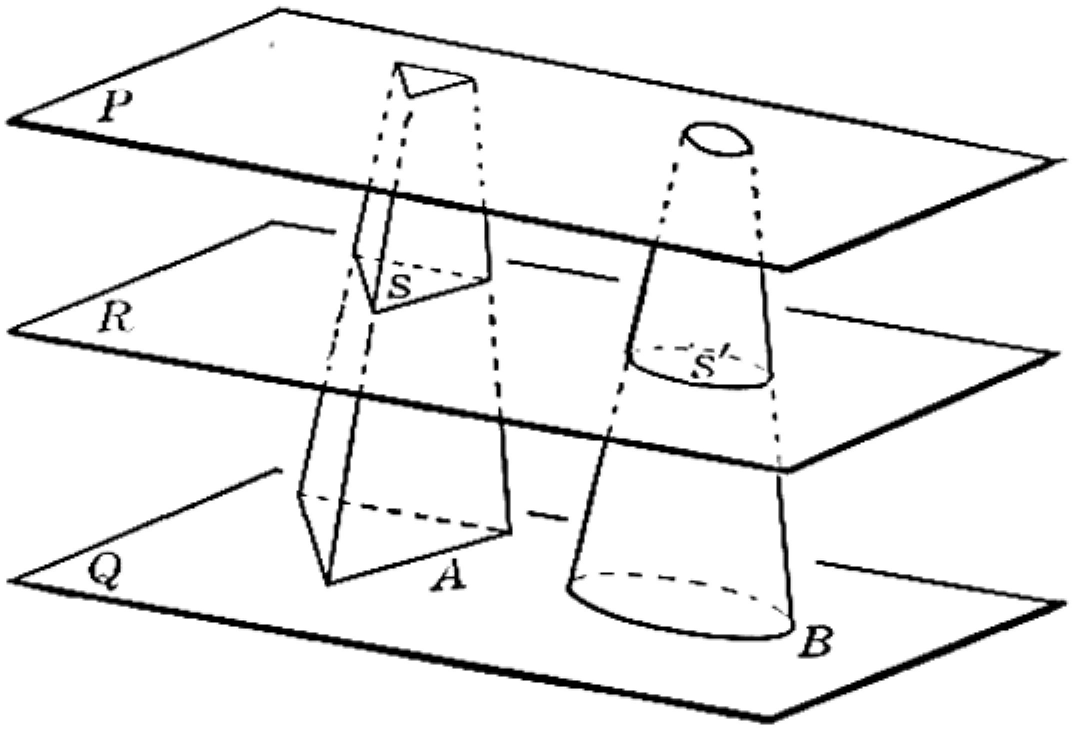
175. Suppose that the solid A in the figure is built up by stacking a large number of cards all uniformly thin. The solid B is similarly built up from a number of disks, each disk having the same thickness as the cards in stack A. Let us assume that the area of the first card in solid A equals the area of the first disk in solid B; the area of the second card equals the area of the second disk; and so on up. It is evident that the two solids have equal volumes.



By increasing indefinitely the number of slices in each solid, thereby decreasing their thickness, we arrive at a basic proposition which was first discovered by the Italian mathematician Bonaventura Cavalieri. The proof of this proposition involves a more detailed study of limits than we have had. We shall therefore assume it without proof.

Note. In 1629 Cavalieri announced the general "theory of indivisibles." This states that a straight line is made up of an infinite number of points, a plane of an infinite number of lines, and a solid of an infinite number of planes.

176. Postulate. *If two solids lie between two parallel planes, and the two sections of the solids made by any plane parallel to these planes are equal in area, then the solids are equal in volume (Cavalieri's Proposition).*



HYPOTHESIS: Solids A and B lie between parallel planes P and Q. R, any plane $\parallel$ to P, makes $=$ sections S and S' on A and B.

CONCLUSION: $A = B$.

To think about Cavalieri's proposition dynamically, imagine that two solids are completely immersed in a tank of water. As the water is being drained from the tank, the solids will intersect the surface of the water in two sections. If the areas of these two sections are equal, no matter what the level of the water, then the solids have equal volumes.

Cavalieri's proposition may therefore be stated as follows:

If the two areas that a translated plane cuts from two solids are always equal, the solids are equal.

If two solids satisfy the conditions of Cavalieri's proposition, we shall refer to them as Cavalieri solids.

The frontispiece of this book is a photograph of a student's model of Cavalieri solids made by pasting together the same number of similar rectangular cards and circular cards. To make such a model proceed as follows.

Suppose that you wish the dimensions of your rectangles to have the ratio of 5 : 3. Represent these dimensions by $5x$ and $3x$, where x will take on varying values depending on the scale of your drawings. To find the radius r of the corresponding disk, we have

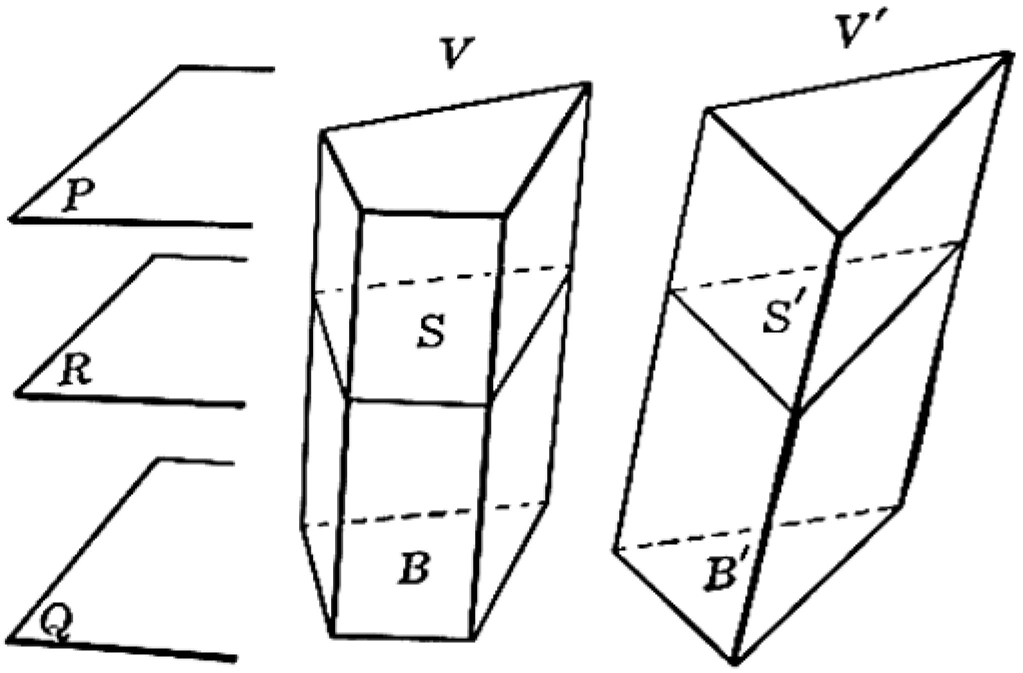
$$\pi r^2 = 5x \cdot 3x = 15x^2, \qquad r^2 = \frac{15x^2}{\pi} = 4.78x^2, \qquad r = 2.19x$$

Now make a table of suitable values for x and the corresponding values of $5x$, $3x$, and $2.19x$. These are the sizes of your cards and disks.

Equal Prisms

177. If two prisms have equal altitudes, we can place them so that their lower bases are in one plane and their upper bases are in another plane parallel to the lower plane. If these prisms also have equal bases, we can show that the sections made by *any* plane parallel to their bases are equal. This makes the prisms Cavalieri solids and therefore equal in volume.

178. THEOREM. Two prisms having equal bases and equal altitudes are equal.



GIVEN: Prisms V and V' with = bases B and B' and = altitudes.

PROVE: $V = V'$.

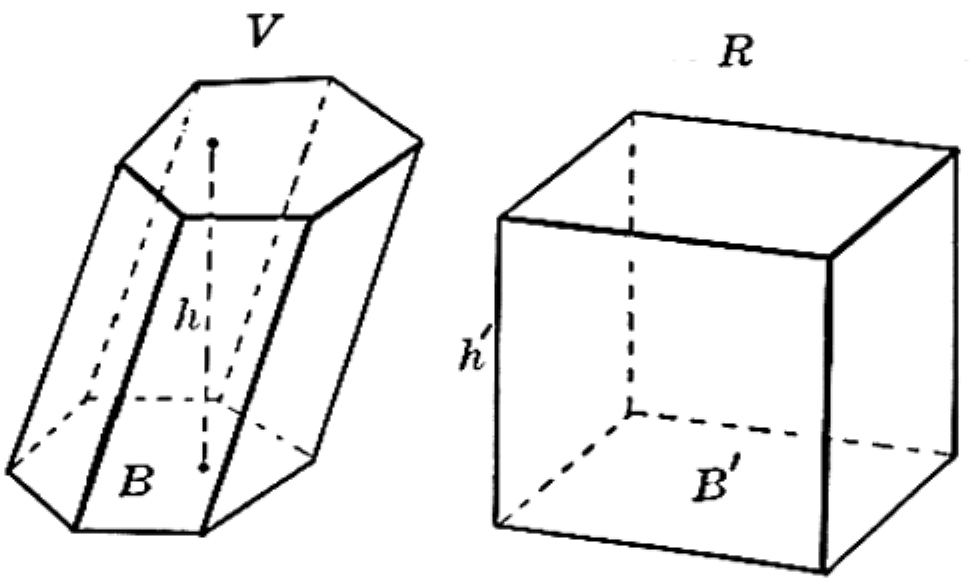
CONS.: Since their alts. are =, place the prisms with their bases in ∥ planes P and Q.
Draw *any* third plane R ∥ to P, cutting V in S and V' in S'.

PROOF: *Statements*	*Reasons*
1. $S = B$, $S' = B'$.	1. The sections of a prism formed by two ∥ planes cutting all the edges are ≅. (§ 130)
2. Since $B = B'$,	2. Given.
3. $S = S'$.	3. Subs. Ax.
4. $V = V'$.	4. Cavalieri's Proposition. (§ 176)

The Volume of a Prism and a Cylinder

179. To find the volume of any prism, we draw a rectangular solid that has its base and altitude equal respectively to the base and altitude of the given prism. Since the volume of the rectangular solid equals the product of its base and altitude, the volume of the given prism is also equal to the product of its base and altitude by substitution.

180. THEOREM. The volume of any prism equals the product of its base and altitude.



Given: Prism V with base B and altitude h.

Prove: $V = Bh$.

Cons.: Draw rectangular solid R with base $B' = B$, and altitude $h' = h$.

Write out the proof.

Since this theorem does not depend on the number of lateral faces of the prism, we may apply it to a cylinder (§ 147) and obtain the following corollary.

181. Corollary. The volume of a cylinder equals the product of its base and altitude.

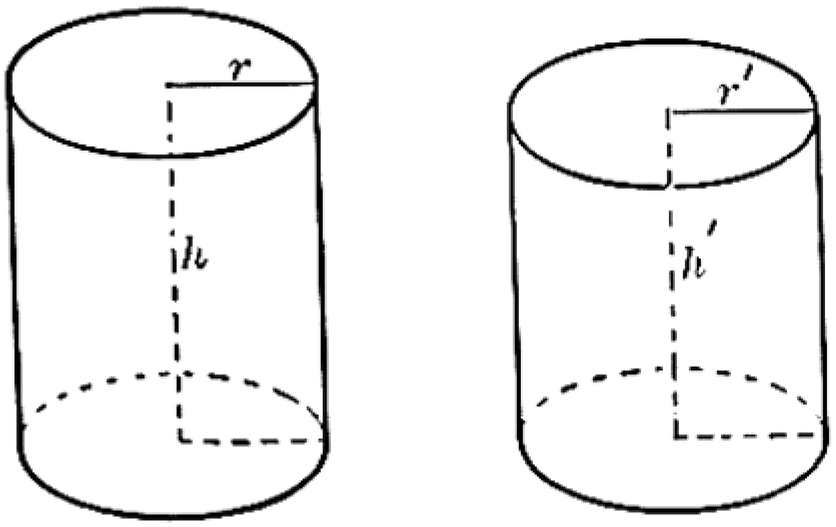
In a circular cylinder $B = \pi r^2$. Therefore $V = Bh = \pi r^2 h$.

182. Corollary. If the volume of a circular cylinder is V, its altitude h, and the radius of the base r, then $V = \pi r^2 h$.

Similar Cylinders of Revolution

183. Definition. *Similar cylinders of revolution are cylinders that are generated by revolving similar rectangles about corresponding sides as axes.*

184. THEOREM. The lateral areas or the total areas of two similar cylinders of revolution are proportional to the squares of their altitudes or to the squares of the radii of their bases; their volumes are proportional to the cubes of their altitudes or to the cubes of the radii of their bases.



GIVEN: In the similar cylinders of revolution
L and L' the lateral areas,
T and T' the total areas,
V and V' the volumes,
h and h' the altitudes,
r and r' the radii of the bases.

PROVE: $\frac{L}{L'} = \frac{T}{T'} = \frac{h^2}{h'^2} = \frac{r^2}{r'^2}$.
$\frac{V}{V'} = \frac{h^3}{h'^3} = \frac{r^3}{r'^3}$.

Supply the reason for each statement in the proof.

PROOF:

1. $\frac{r}{r'} = \frac{h}{h'}$.
2. $\frac{r+h}{r'+h'} = \frac{r}{r'} = \frac{h}{h'}$. (§ 548)
3. $L = 2\pi rh$.
   $L' = 2\pi r'h'$.
4. $\frac{L}{L'} = \frac{rh}{r'h'} = \frac{r}{r'}\cdot\frac{h}{h'} = \frac{r}{r'}\cdot\frac{r}{r'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}$.
5. $T = 2\pi r(r+h)$.
   $T' = 2\pi r'(r'+h')$.
6. $\frac{T}{T'} = \frac{r(r+h)}{r'(r'+h')} = \frac{r}{r'}\cdot\frac{r+h}{r'+h'} = \frac{r}{r'}\cdot\frac{r}{r'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}$.
7. $V = \pi r^2 h$.
   $V' = \pi r'^2 h'$.
8. $\frac{V}{V'} = \frac{r^2h}{r'^2h'} = \frac{r^2}{r'^2}\cdot\frac{h}{h'} = \frac{r^2}{r'^2}\cdot\frac{r}{r'} = \frac{r^3}{r'^3} = \frac{h^3}{h'^3}$.

EXERCISES

1. Find the cost of excavating a rectangular cellar 40 feet long, 25 feet wide, and 6 feet deep at $9 per cubic yard.

2. Find the volume of a cube if its total area is 150 square inches.

3. The sum of the volumes of two cubes is 1064 cubic inches. Find the edge of each cube if they are in the ratio 2:5.

4. The volume of a rectangular solid is 324 cubic inches. Find the dimensions if they are in the ratios 1:3:4.

5. The number of cubic inches in the volume of a cube is twice the number of square inches in its total surface. Find the edge of the cube.

6. Find the volume of a cube whose diagonal is 3 inches.

7. The unit of lumber measure is the *board foot.* A board foot is a board one foot square and one inch or less in thickness. If the thickness of a board is less than one inch, it has the same number of board feet as if the thickness were one inch.

Example 1. How many board feet are in a piece of lumber $9'' \times \frac{5}{8}'' \times 12'$?

$$12 \times \tfrac{9}{12} \times 1 = 9 \text{ bd. ft.}$$

Example 2. How many board feet are in a beam $6'' \times 2\frac{5}{8}'' \times 16'$?

$$16 \times \tfrac{6}{12} \times 2\tfrac{5}{8} = 21 \text{ bd. ft.}$$

Find the number of board feet of lumber in a piece $4'' \times 1\frac{1}{2}'' \times 14'$.

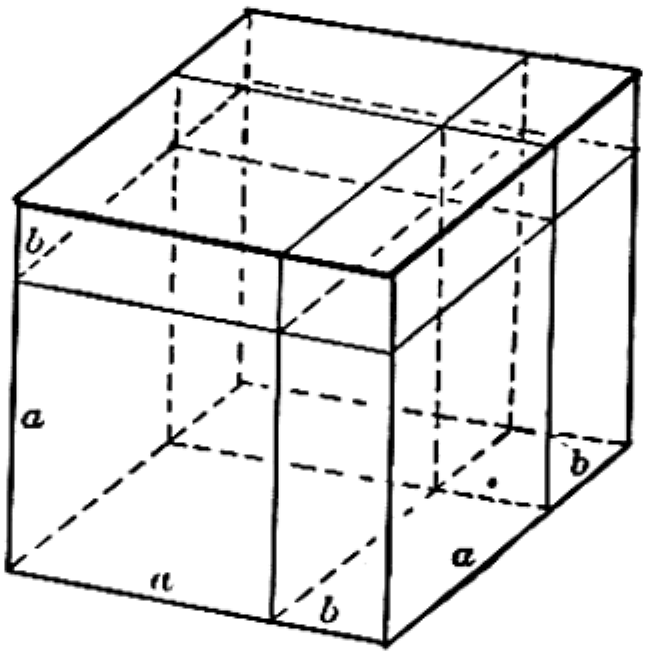
8. Find the cost at $160 per M (M is 1000 board feet) of the following lumber:

6 pieces $2'' \times 4'' \times 8'$
3 pieces $2'' \times 6'' \times 14'$
8 pieces $\frac{1}{2}'' \times 4'' \times 12'$

9. The diagram shows a cube whose side is $a + b$.

   a. Find the areas of the squares and rectangles on one face of the cube and show that it is the same result as you get in algebra when you square the binomial $a + b$.

   b. By finding the volumes of the rectangular solids in the figure, derive the formula for $(a + b)^3$.



10. Find the ratio of the volumes of two cubes if an edge of one equals a diagonal of the other.

11. Prove: The plane passed through two diagonally opposite edges of a parallelepiped divides it into two equal prisms.

    What is the analogous proposition in plane geometry?

12. What happens to the volume of a rectangular solid if one dimension is doubled? if two dimensions are doubled? if all three dimensions are doubled? if one dimension is doubled and the other two are trebled?

13. Find the volume of a square prism circumscribed about a cylinder whose diameter is 8 inches and whose altitude is 6 inches.

14. A cylindrical steel bar 2 inches in diameter and 10 inches long is to be machined to make the largest possible square shaft. Find the volume of the finished shaft.

15. The base of a prism is a rhombus with diagonals 12 inches and 7 inches. Find the volume of the prism if its altitude is 20 inches.

16. A round pole 12 inches in diameter and 8 feet long is made into a hexagonal column. Find the largest possible volume of the column.

17. Find the volume of a regular octagonal prism if its altitude is 8 feet and the radius of its base is 6 inches.

18. Find the volume of a right triangular prism in which each edge is a.

19. Complete the following statements.

a. If two prisms or cylinders have equal bases, they are proportional to their —?—.

b. If two prisms or cylinders have equal altitudes, they are proportional to their —?—.

c. If a cylinder is inscribed in a square prism, the ratio of the volumes of the cylinder and the prism is —?—.

20. The altitude of a regular triangular prism is $4\sqrt{3}$ inches and its volume is 108 cubic inches. Find one side of the base.

21. Prove that the volume of a triangular prism equals half the product of the area of one lateral face and the length of the perpendicular to that face from the opposite lateral edge. *Hint*: Complete the parallelepiped.

22. Find the volume of a cylinder inscribed in a cube whose edge is e.

23. The dimensions of a rectangle are a and b. Find the ratio of the volumes of the cylinders generated by rotating the rectangle about either side as an axis.

24. In two similar cylinders of revolution:

a. if the altitudes are in the ratio 3:5, find the ratio of the lateral areas, total areas, and volumes.

b. if the lateral areas are in the ratio 4:9, find the ratio of the radii, total areas, and volumes.

c. if the volumes are in the ratio 1:8, find the ratio of their total areas.

25. A cylindrical tank is 12 feet in diameter. How long will it take to fill the tank to a depth of 8 feet if water flows in at the rate of 100 gallons per minute? (1 cu. ft. = 7.48 gal.)

26. A cylindrical tank 6 feet in diameter is filled to a depth of 4 feet. It is being pumped dry by a 2-inch pipe. If the liquid flows through the pipe at 30 feet per minute, how long will the pumping take?

27. A rectangular bar of metal 1 inch by 3 inches and 12 inches long is recast into a cylindrical shape of 2-inch diameter. How long is the recast bar?

28. A cylindrical can is to have a capacity of one quart. What should be the height of the can if its diameter is 4 inches? (1 gal. = 231 cu. in.)

29. Each edge of a regular triangular prism is 12 inches. Find the volume of the inscribed cylinder.

30. Find the ratio of the volumes of the cylinders inscribed in and circumscribed about the same regular triangular prism.

31. If the plane section containing the axis of a right cylinder is equilateral, find the lateral area, the total area, and the volume of the cylinder in terms of its diameter d.

32. An aluminum casting is in the form of a regular hexagonal prism. Its altitude is 12 inches and a side of its base is 4 inches. If a 2-inch hole is bored from base to base, find the volume of the remaining aluminum.

33. The lateral area of a right cylinder is 100 square inches and its altitude is 5 inches. Find the volume of the cylinder.

34. How many feet of wire $\frac{1}{32}$ inch in diameter can be drawn from one cubic foot of copper?

35. A crystal placed in a graduate of water causes the water to rise 1.5 centimeters. If the diameter of the graduate is 2 centimeters, what is the volume of the crystal?

36. A cylindrical tank is resting on its base, which is 2 feet in diameter. It is desired to calibrate a measuring stick in gallons. How many inches apart should the gallon marks be placed?

37. Derive the formula for the volume of a right cylinder in terms of its lateral area L and its altitude h.

38. Prove that the volume of a regular prism equals half the product of its lateral area and the apothem of the base.

39. A swimming pool is 40 feet long and 18 feet wide. Its depth increases along the length from 3 feet to 12 feet. Find the capacity of the pool in gallons. *Hint:* Consider the pool as a prism with vertical bases.

40. A railroad cut is to be 150 feet long and 9 feet deep. A cross section of the cut is to be a trapezoid 14 feet wide at the bottom and 22 feet wide at the top. Find the cost of removing the earth at $6 a cubic yard.

SUPERIOR WORK

S 1. The base of a prism is a triangle with sides 13, 14, and 15 inches. One lateral edge is 20 inches and its projection on the plane of the base is 12 inches. Find the volume of the prism.

S 2. The radius of the base of a cylinder is increased 20 per cent and the altitude is increased 30 per cent. By what per cent is the volume increased?

S 3. A rectangular solid 5 feet long has a volume of 60 cubic feet and a total area of 94 square feet. Find the other dimensions.

S 4. A plane bisects a dihedral angle formed by two lateral faces of a triangular right prism. Prove that the prism is divided into two parts whose volumes have the same ratio as the areas of the faces of the bisected dihedral angle.

S 5. The base of a right prism is a triangle with sides 5, 12, and 13 inches. A plane is drawn perpendicular to the 13-inch side and passing through the opposite lateral edge of the prism. If the altitude of the prism is 10 inches, find the volumes of the two parts into which the prism is divided.

S 6. Two lateral faces of a triangular prism are rectangles 8 inches by 6 inches, and 8 inches by 5 inches. The angle between these faces is 60°. Find the volume of the prism.

S 7. Every edge of a regular octagonal prism is 12 inches. Find the volume of the prism. (Use trigonometric functions.)

S 8. The base of a parallelepiped is a parallelogram having a 40° angle and sides 12 inches and 8 inches. One lateral edge of the parallelepiped is 10 inches long and makes an angle of 70° with the plane of the base. Find the volume of the parallelepiped. (Use trigonometric functions.)

S 9. Two prisms or cylinders rest on the same base and have equal elements which are differently inclined to the plane of the base. Prove that the volumes of the two solids are proportional to the sines of the inclinations.

S 10. A cylindrical tank 4 feet in diameter and 7 feet long rests on its cylindrical surface. How many gallons of oil does it contain when filled to a depth of 2.75 feet? (Use trigonometric functions.)

SUMMARY

185. Prisms and cylinders are formed in part by ruled surfaces that are generated by translating a line. By using the propositions of the first two chapters, we derived a number of properties of a prism. We then assumed that any proposition about a prism with regular bases which does not depend on the number of lateral faces may be applied to a cylinder.

Cavalieri's proposition, which we assumed, is useful in comparing the volumes of two solids. It states that if the two areas that a translated plane cuts from two solids are always equal, the solids are equal.

We proved the following important theorems:

A plane section of a prism or a cylinder which contains an element is a parallelogram.

The lateral area of a prism or a cylinder equals the product of an element and the perimeter of a right section.

The volume of a prism or a cylinder equals the product of its base and altitude.

In two similar cylinders of revolution the lateral areas or the total areas have the same ratio as the squares of the altitudes or the squares of the radii of the bases; but the volumes have the same ratio as the cubes of the altitudes or as the cubes of the radii of the bases.

RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, go back to the section indicated.

- translated line or plane § 115
- ruled surface § 116
  - generatrix § 116
  - elements § 116
  - directrix § 116
- prismatic surface § 117
- cylindrical surface § 117
- closed prismatic surface § 118
- closed cylindrical surface § 118
- prism § 119
  - bases § 120
  - lateral surface § 120
  - lateral faces § 121
  - elements § 120
  - lateral edge § 122
  - altitude § 123
  - right section § 124
- cylinder § 119
  - bases § 120
  - element § 120
  - lateral surface § 120
  - altitude § 123
  - right section § 124
- right prism § 125
- right cylinder § 125
- oblique prism § 125
- oblique cylinder § 125
- triangular prism § 126
- quadrangular prism, etc. § 126
- circular cylinder § 127
  - axis § 127
- cylinder of revolution § 138
- plane tangent to cylinder § 139
- prism circumscribed about cylinder § 140
- prism inscribed in cylinder § 141
- regular prism § 142
- limit of variable § 145
- parallelepiped § 153
- right parallelepiped § 159
- rectangular solid § 160
  - dimensions § 161
- cube § 162
- truncated prism § 167
- unit of volume § 169
- volume § 170
- equal solids § 171
- Cavalieri's proposition § 176
- Cavalieri solids § 176
- similar cylinders of revolution § 183

REVIEW EXERCISES

1. Which of the following statements are true and which are false?

- **a.** Prisms and cylinders are special kinds of polyhedrons.
- **b.** In a prism every lateral edge is an element, and conversely.
- **c.** It is possible for a plane section of an oblique cylinder to be a rectangle.
- **d.** A plane section of a hexagonal prism is not necessarily a hexagon.
- **e.** In a parallelepiped the bases may be considered as lateral faces of a prism.
- **f.** Any plane section of a rectangular solid that cuts all the lateral edges is a rectangle.
- **g.** Although a variable steadily increases, it may not approach a limit.
- **h.** Any proposition about a regular hexagonal prism is also true for a cylinder.
- **i.** If a prism and a cylinder have equal altitudes, they may be made to lie between the same two parallel planes.
- **j.** If two solids lie between the same two parallel planes, they are Cavalieri solids.

2. Complete the following statements:

- **a.** A prism must have at least —?— faces.
- **b.** A quadrangular prism has —?— diagonals.
- **c.** A triangular prism has —?— diagonals.
- **d.** The sum of the face angles of a triangular prism equals —?— degrees.
- **e.** In a right prism the lateral faces are all —?—.
- **f.** The part of an oblique prism between two right sections is a(n) —?— prism.
- **g.** A pentagonal prism may be divided into —?— or more triangular prisms.
- **h.** Any plane section of a prism is a(n) —?—.
- **i.** In a right parallelepiped, at least —?— faces are rectangles.
- **j.** A right circular cylinder is also called a(n) —?—.

3. Prove that the opposite dihedral angles of a parallelepiped are equal.

4. Find the ratio of the area of a plane section through two opposite edges of a cube to the total surface of the cube.

5. At 2 cents a square foot a painter charged $22.80 to paint the walls and ceiling of a room 20 feet by 15 feet. How high is the room?

6. Find the ratios of the three cylinders that can be circumscribed about a rectangular solid whose dimensions are 6 inches by 4 inches by 3 inches.

7. The base of a right prism is a rhombus having one side 10 inches and one angle 60°. If the altitude of the prism is 12 inches, find the total area and volume of the inscribed cylinder.

8. A round bushel basket has its height equal to its diameter. Find its diameter in terms of a radical. (1 bu. = 2150 cu. in.)

9. The diameter of a right circular cylinder is 8 feet. From a point 8 feet from the axis of the cylinder two planes are drawn tangent to the cylinder. Find the dihedral angle between the planes.

10. The base of a right prism is a triangle with sides 6, 8, and 10 inches. The altitude of the prism is 5 inches. Find the difference between the volumes of the circumscribed and inscribed cylinders.

11. Prove that the volume of a regular triangular prism in which each edge is a equals $\frac{a^3}{4}\sqrt{3}$.

12. The altitude of a right prism is 15 inches. Each base is a trapezoid with bases 8 inches and 18 inches, and legs 13 inches each. Find the total area and the volume of the prism.

13. The base of an oblique prism is an equilateral triangle of side 10 inches. A lateral edge of the prism is 12 inches, and the volume of the prism is 450 cubic inches. Find the inclination of the lateral edges to the plane of the base.

14. From the following data find the number of quarts in a *liter:*

$$\begin{aligned} 1 \text{ liter} &= 1 \text{ cubic decimeter} \\ 1 \text{ meter} &= 10 \text{ decimeters} = 39.37 \text{ inches} \\ 1 \text{ gallon} &= 4 \text{ quarts} = 231 \text{ cubic inches} \end{aligned}$$

15. The number of square inches in the total area of a cube exceeds the number of inches in the edge of the cube by 15. Find the edge of the cube.

16. The lateral area of a cylinder of revolution is 80 square inches, and the total area is 180 square inches. Find the volume of the cylinder.

17. The altitude of a right cylinder is 10 and its total area is 112π. Find the area of the base.

18. Prove that the volume of a right cylinder equals half the product of the lateral area and the radius of the cylinder.

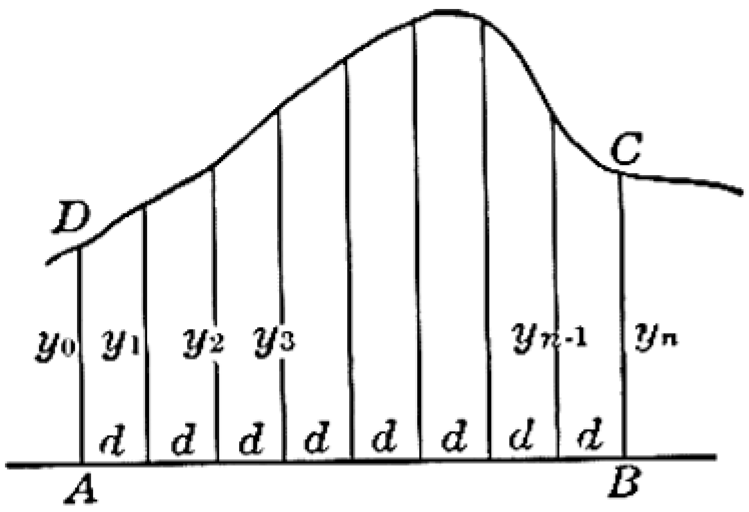
19. The number of cubic feet in the volume of a right cylinder is twice the number of square feet in its total area. Find the volume of the cylinder if its altitude equals its radius.

20. The base of an oblique triangular prism is a right triangle with legs 9 inches and 12 inches. The area of the largest lateral face is 240 square inches. The base makes an angle of 60° with the largest lateral face. Find the volume of the prism.

21. Solve Exercise 20 if the base makes an angle of 70° with the largest lateral face. (Use trigonometric functions.)

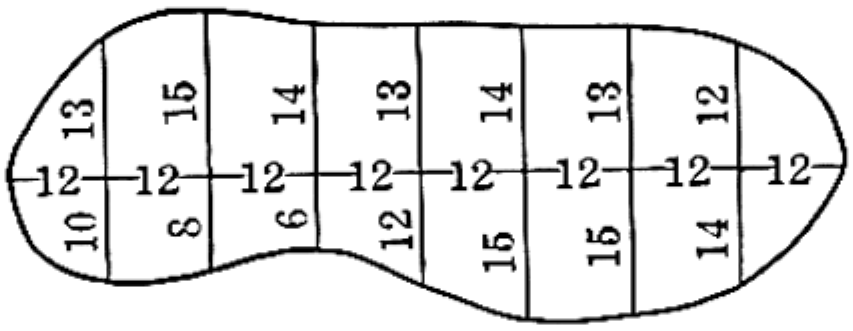
SUPERIOR WORK

The Trapezoidal Rule



S 186. To measure an irregular area *ABCD* between a line and a curve, surveyors divide *AB* into equal parts and measure the perpendiculars y_0, y_1, y_2, ... By assuming that the figure is divided into trapezoids, we can derive by plane geometry the following formula known as the *trapezoidal rule*:

$$\text{Area } ABCD = d\left(\frac{y_0 + y_n}{2} + y_1 + y_2 + y_3 + \cdots + y_{n-1}\right)$$



S 1. The irregular curve in the figure is the base of a cylinder whose altitude is 50. Using the measurements given in the figure and the trapezoidal rule, find the volume of the cylinder. (In this case $y_0 = 0$, and $y_8 = 0$.)

S 2. Soundings at intervals of 3 feet across a stream were found to be as follows: 7.1, 14.3, 16.5, 17.2, 17.0, 16.1, 12.9, and 5.4 feet. If the average rate of the stream is 20 feet per minute, how many gallons of water per hour can be expected to flow over a projected dam?

TEST

1. A classroom is 24 feet by 18 feet by 12 feet. It holds 25 pupils. If each pupil requires 30 cubic feet of air per minute, how often, to the nearest minute, should the air in the room be changed?

2. If the diagonal of one face of a cube is $10\sqrt{6}$ inches, find the diagonal of the cube.

3. Express the volume of a cube in terms of its diagonal d.

4. The base of a prism is a rhombus having an angle of 135° and a side 10 inches. Find the volume of the prism if its altitude is 12 inches.

5. How much will it cost to paint the cylindrical wall of a silo 20 feet high and 15 feet in diameter at 10 cents a square foot?

6. The radius of a right circular cylinder is 10 inches. The total area is 280π square inches. Find the altitude of the cylinder.

7. Find the ratio of the volume of a regular quadrangular prism to the volume of the circumscribed cylinder.

8. The base of a right prism is a triangle with sides 6, 8, and 10 inches. If the total area of the prism is 168 square inches, find its volume.

9. Which of the following statements are true and which are false?

   a. A plane section of a regular prism cutting all the lateral edges is a regular polygon.
   b. If two prisms have equal bases and equal altitudes, they have equal lateral areas.
   c. The volume of a parallelepiped equals the product of any face and the altitude to that face.
   d. The volume of a right cylinder equals the product of the base and an element.
   e. A right parallelepiped is the same as a rectangular parallelepiped.
   f. If the lateral faces of a right parallelepiped are rectangles, the figure is not necessarily a rectangular solid.
   g. The lateral areas of two similar cylinders of revolution have the same ratio as the total areas.
   h. A cylinder cannot be formed by revolving a parallelogram about one side as an axis.

10. Complete each of the following statements:

   a. If the axis of a cylinder is an altitude, the figure is a(n) —?— cylinder.
   b. An element of a cylindrical surface is a straight line, but an element of a cylinder is a(n) —?—.
   c. The lateral area of a prism equals the product of a lateral edge and —?—.
   d. Every dihedral angle of a regular hexagonal prism is either —?— or —?— degrees.
   e. If the volume of one cube is twice the volume of another cube, the ratio of their edges is —?—.
   f. If the edge of a cube is 6 units, the ratio of the number that expresses its volume to the number that expresses its area is —?—.
   g. If the lateral areas of two similar cylinders of revolution are 17 square inches and 163 square inches, the ratio of their volumes is —?—.
   h. If the radius of a circular cylinder is doubled, its volume is multiplied by —?—.

4 Rotating a Line

Properties of Pyramids and Cones

FORMATION OF PYRAMIDS AND CONES

187. From the preparatory exercise on page 82 you may have discovered that the curved surface of a cone is a ruled surface. Review the ideas connected with a ruled surface in § 116. The lateral surfaces of pyramids and cones are formed in the same way as those of prisms and cylinders, except that the ruled surfaces (§ 116) involved are generated by rotating instead of translating a line.

By referring to the diagram on the next page, trace the formation of the pyramid. In the first diagram the generatrix rotates about a fixed point V, called the *vertex,* always touching a fixed directrix d in another plane. This forms the open *pyramidal surface* shown below. The various positions of the generatrix are called the elements of the pyramidal surface. The parts of the surface on each side of V are called its *nappes.* The lowest diagram is a solid bounded by one nappe of a closed pyramidal surface and a plane. This is a *pyramid.*

In like manner, trace the formation of the cone.

Definitions

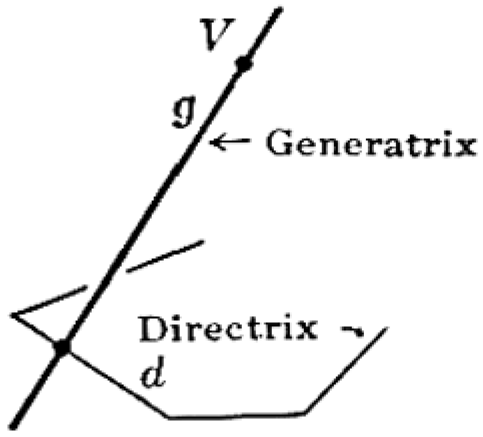
188. *A pyramidal surface is a ruled surface in which the generatrix is rotated about a point and the directrix is a broken line. A conical surface is a ruled surface in which the generatrix is rotated about a point and the directrix is a curve.*

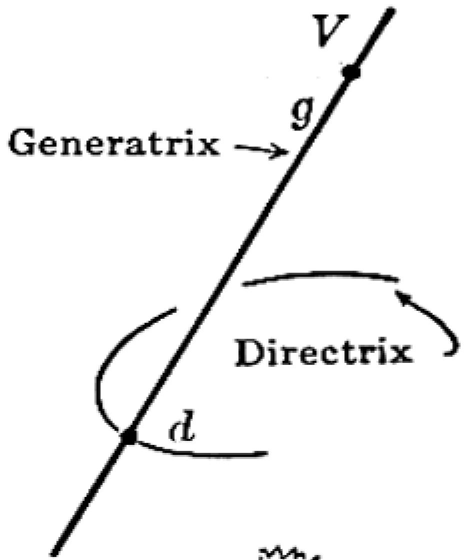
189. *The parts of the pyramidal or conical surface on each side of the point about which the generatrix rotates are called the nappes of the surface.*

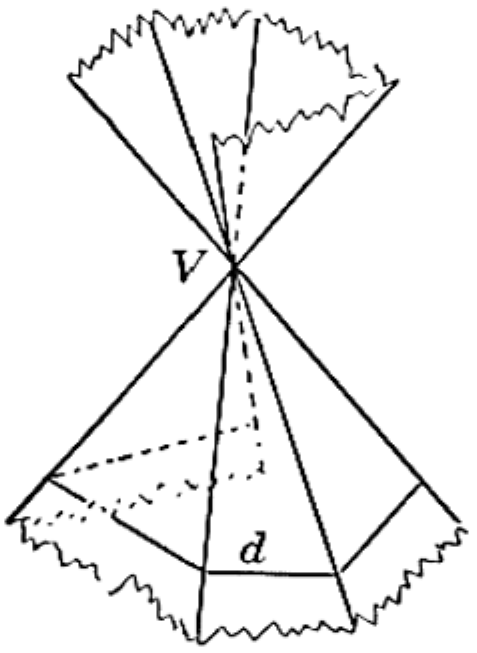
190. *A closed pyramidal or conical surface is one in which the directrix is closed.*

191. *A pyramid or a cone is a solid bounded by one nappe of a closed pyramidal or conical surface and a plane cutting all the elements.*

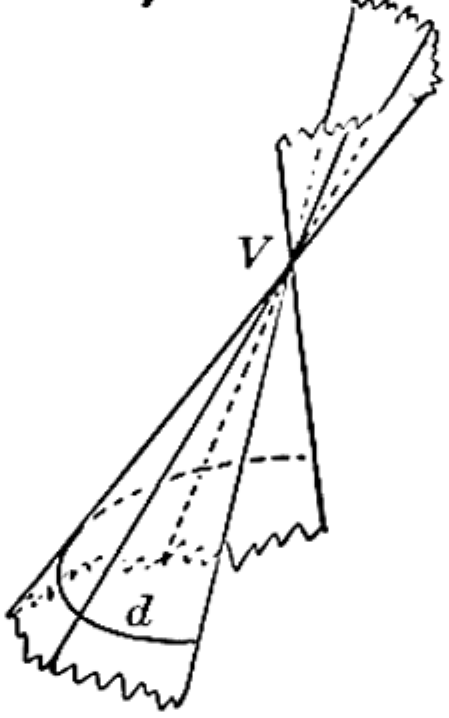
192. *In a pyramid or a cone the point about which the generatrix rotates is the vertex, the part of the plane bounded by the pyramidal or* *conical surface is the base, the part of the pyramidal or conical surface included between the base and the vertex is the lateral surface, the parts of the elements of the pyramidal or conical surface between the vertex and the base are the elements of the pyramid or the cone.*



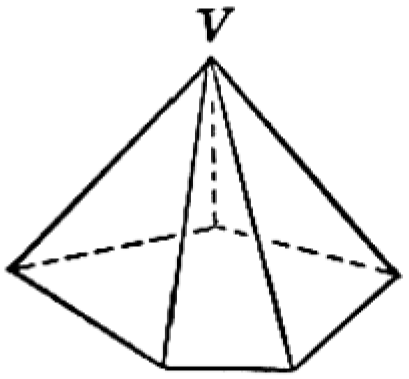




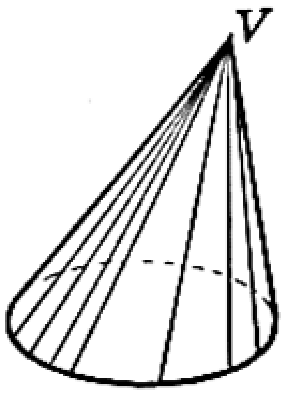
Pyramidal Surface



Conical Surface



Pyramid



Cone

193. *The faces of a pyramid other than the base are the lateral faces.*

194. *An element of a pyramid that passes through a vertex of the base is called a lateral edge.*

195. *The altitude of a pyramid or a cone is the length of the perpendicular from the vertex to the base.*

196. *A pyramid is triangular, quadrangular, pentagonal, etc., according as its base is a triangle, a quadrilateral, a pentagon, etc.*

197. *A circular cone is a cone in which the base is a circle. The line segment joining the vertex to the center of the base is called the axis of the cone.*

Unless otherwise stated, we shall refer to a circular cone simply as a cone.

198. *A regular pyramid is a pyramid whose base is a regular polygon and whose altitude passes through the center of the base. A right circular cone is one whose altitude passes through the center of the base; otherwise the cone is oblique.*

The altitude of a regular pyramid is also called its axis.

199. *The slant height of a regular pyramid is an element of the pyramid that is perpendicular to an edge of the base. The slant height of a right circular cone is any element of the cone.*

EXERCISES

1. **a.** Why is a plane section of a pyramid a polygon?
  b. Why is every section of a pyramid made by a plane passing through its vertex a triangle?
  c. What is the least number of faces that a pyramid may have? What is the name of this pyramid? What kind of polyhedron is this pyramid?

2. **a.** Do all pyramids have slant heights? How many slant heights does a pyramid have?
  b. Do all cones have slant heights? How many slant heights does a cone have?

3. Prove: **a.** The lateral edges of a regular pyramid are equal.
  b. The slant heights of a regular pyramid are equal.
  c. The elements of a right circular cone are equal.

4. **a.** In what kind of pyramid are the lateral faces congruent? What kind of figures are these faces?
  b. Prove that the lateral edges of a regular pyramid make equal angles with the base.
  c. Prove that the slant heights of a regular pyramid make equal angles with the base.
  d. Prove that the elements of a right circular cone make equal angles with the base.

5. **a.** Draw a pattern, cut it out, fold and paste it to form a regular triangular pyramid.
  b. Is your model a regular tetrahedron?
  c. Check Euler's formula $E + 2 = V + F$ (page 71, Exercise 8) in your model.
  d. Check the sum of the face angles in your model with the formula

$$S = (V - 2)2 \text{ st. } \measuredangle\text{s (page 73, S 1).}$$

6. Prove that if a right triangle is rotated about one leg as an axis, it generates a right circular cone.
7. The altitude of a right circular cone is 12 inches and the radius of the base is 9 inches. Find the slant height.
8. The slant height of a regular hexagonal pyramid is 12 inches, and one side of the base is 10 inches. Find the altitude of the pyramid.
9. The area of the base of a right circular cone is 64π square inches, and the slant height is 10 inches. Find the altitude of the cone.
10. Find the altitude of a regular tetrahedron whose edge is 10 inches.

PROPERTIES OF THE PYRAMID AND THE CONE

In the preceding exercises you proved certain properties of pyramids and cones. Some of these are reviewed in the following corollaries.

Corollaries

200. The lateral faces of a pyramid are triangles.

201. The slant heights of a regular pyramid or a right circular cone are equal.

202. The lateral faces of a regular pyramid are congruent isosceles triangles.

203. A right circular cone may be generated by revolving a right triangle about one leg as an axis.

204. The altitude of a right circular cone and its axis are the same line.

205. Definition. *A cone of revolution is a right circular cone.*

Just as we have associated the cylinder with inscribed and circumscribed prisms in the preceding chapter, we shall now associate the cone with inscribed and circumscribed pyramids.

Definitions

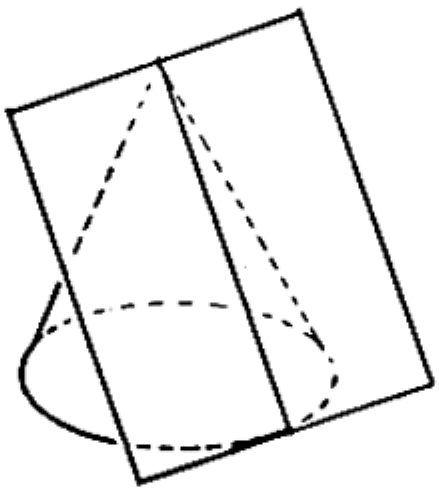
206. *A plane is tangent to a cone if it contains only one element of the cone.*

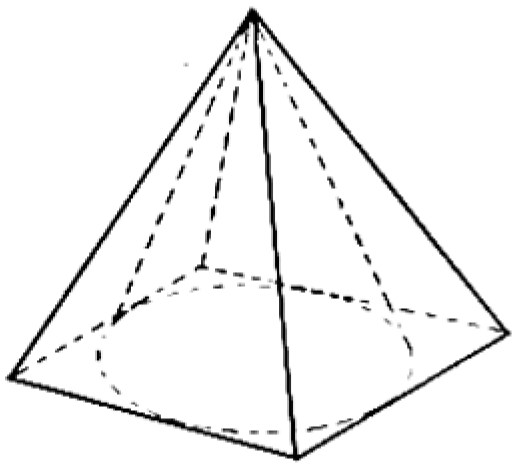
207. *A pyramid is circumscribed about a cone if its vertex is the same as the vertex of the cone and its base is circumscribed about the base of the cone.*

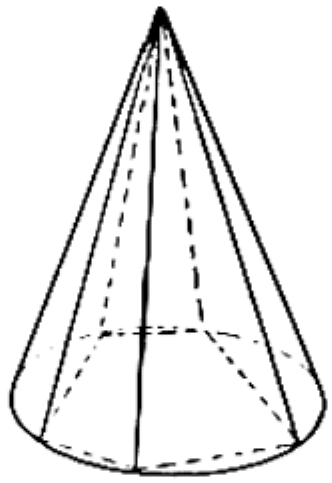
208. Corollary. The faces of a pyramid circumscribed about a cone are tangent to the cone.

209. Definition. *A pyramid is inscribed in a cone if its vertex is the same as the vertex of the cone and its base is inscribed in the base of the cone.*

210. Corollary. The lateral edges of a pyramid inscribed in a cone are elements of the cone.


Plane Tangent
to Cone


Pyramid Circumscribed
about Cone


Pyramid Inscribed
in Cone

211. Corollary. A pyramid with a regular base can be inscribed in or circumscribed about a given cone.

212. Corollary. If the number of lateral faces of an inscribed pyramid with a regular base is repeatedly doubled, the limit of the pyramid is the cone in which it is inscribed.

213. Postulate. *Any proposition about a pyramid with a regular base that does not depend on the number of lateral faces is equally valid for the circumscribed cone.*

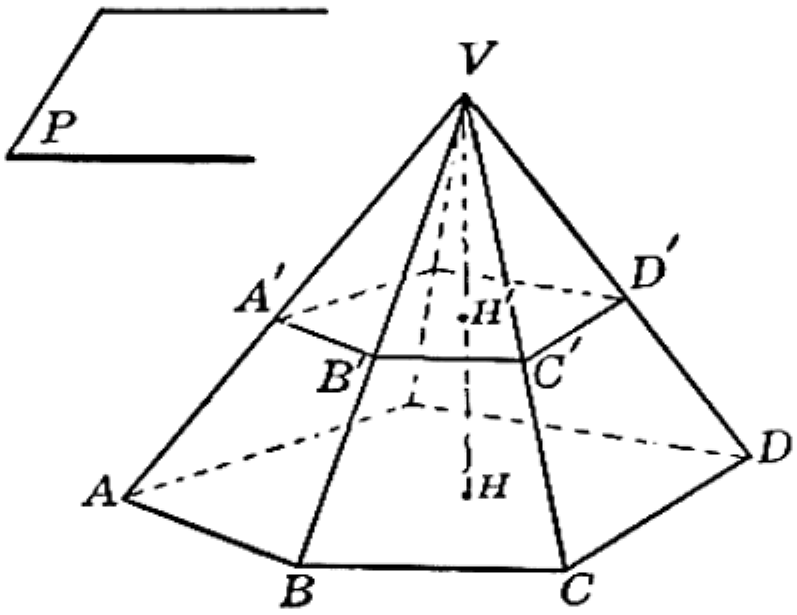
In Exercise 1 on page 113, you proved that every section of a pyramid made by a plane passing through the vertex is a triangle. The preceding postulate permits us to apply this to a cone.

214. Corollary. Every section of a pyramid or a cone made by a plane passing through the vertex is a triangle.

215. Definition. *An equilateral cone is one whose axis section is an equilateral triangle.*

We shall now study the relationships that result when a pyramid is cut by a plane parallel to the base.

216. THEOREM. If a pyramid is cut by a plane parallel to the base: (a) the lateral edges and the altitude are divided proportionally; (b) the section is a polygon similar to the base; (c) the area of the section is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.



GIVEN: Pyramid $V\text{–}ABCD\cdots$ cut by a plane $\parallel$ to its base, intersecting the lateral edges in A', B', C', D', . . . and the altitude VH in H'.

PROVE:

$$\frac{VA'}{VA} = \frac{VB'}{VB} = \frac{VC'}{VC} = \cdots = \frac{VH'}{VH}. \qquad (a)$$

$$A'B'C'D'\cdots \sim ABCD\cdots. \qquad (b)$$

$$\frac{A'B'C'D'\cdots}{ABCD\cdots} = \frac{VH'^2}{VH^2}. \qquad (c)$$

CONS.: Draw plane P through V parallel to $ABCD\cdots$.

PROOF:

Statements	*Reasons*
1. $\frac{VA'}{VA} = \frac{VB'}{VB} = \frac{VC'}{VC} = \cdots = \frac{VH'}{VH}$. (*a*)	1. Three $\parallel$ planes cut lines proportionally. (§ 61)
2. $A'B' \parallel AB$, $B'C' \parallel BC$, ...	2. The intersections of 2 $\parallel$ planes with third plane are $\parallel$. (§ 54)
3. $\angle A'B'C' = \angle ABC$, $\angle B'C'D' = \angle BCD$, etc.	3. If 2 $\angle$s have sides $\parallel$ and in same direction, they are =. (§ 68)
4. $\angle VA'B' = \angle VAB$, $\angle VB'A' = \angle VBA$.	4. $\parallel$s, corr. $\angle$s are =.
5. $\triangle VA'B' \sim \triangle VAB$. Similarly, $\triangle VB'C' \sim \triangle VBC$.	5. If 2$\triangle$s agree in 2$\angle$s, they are $\sim$. (§ 479)
6. $\frac{A'B'}{AB} = \frac{VB'}{VB}$, $\frac{B'C'}{BC} = \frac{VB'}{VB}$.	6. Def. $\sim$ polygons.

7. $\dfrac{A'B'}{AB} = \dfrac{B'C'}{BC}$.	7. Subs. Ax.
Similarly, $\dfrac{A'B'}{AB} = \dfrac{B'C'}{BC} = \dfrac{C'D'}{CD} = \cdots.$	
8. $A'B'C'D' \cdots \sim ABCD \cdots$ (b)	8. Def. $\sim$ polygons.
9. $\dfrac{A'B'C'D' \cdots}{ABCD \cdots} = \dfrac{A'B'^2}{AB^2}$	9. $\sim$ polygons are proportional to squares of corr. sides. (§ 495)
10. $\dfrac{A'B'C'D' \cdots}{ABCD \cdots} = \dfrac{VH'^2}{VH^2}$ (c)	10. Subs. Ax.

Since this theorem is true about a pyramid with a regular base and it does not depend on the number of lateral faces, we may apply it to a cone (§ 213). Considering that the regular base and the section parallel to it both approach circles, we have the following.

217. Corollary. If a cone is cut by a plane parallel to the base: (a) the elements and the altitude are divided proportionally; (b) the section is a circle; (c) the area of the section is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the cone.

EXERCISES

1. Prove: If a plane cuts a tetrahedron and is parallel to two skew edges, the section is a parallelogram.

2. Prove: If a section of a tetrahedron made by a plane is a parallelogram, it is parallel to two skew edges of the tetrahedron.

3. Pass a plane through the axis of an oblique circular cone so that the section of the cone is an isosceles triangle. Is there more than one such plane?

4. Pass a plane through the vertex of an oblique circular cone so that the section of the cone is an isosceles triangle. Is there more than one such plane?

5. Prove that if two planes are tangent to the same cone, their intersection passes through the vertex of the cone.

6. Prove: If two planes are tangent to a cone of revolution and they contain the ends of a diameter of the base, the intersection of the planes is parallel to the base.

7. A right triangle whose hypotenuse is 7 inches revolves about its 6-inch leg. Find the area of the base of the resulting cone.

8. If the altitude of a cone of revolution is 6 inches and the slant height is $6\frac{1}{2}$ inches, find the area of the base.

9. A right triangle whose legs are 6 inches and 8 inches revolves about its hypotenuse, forming two cones with a common base. Find the area of the common base.

10. The base of a pyramid is a square 5 inches on a side, and the altitude is 12 inches. Find the area of a section parallel to the base if
    a. the section is 3 inches from the vertex.
    b. the section is 2 inches from the base.
    c. the section is midway between the vertex and the base.
    d. the section is n inches from the base.

11. The base of a pyramid is a triangle with sides 7, 24, and 25 inches. The altitude is 12 inches. Find the area of a section parallel to the base and 9 inches from it.

12. The base of a pyramid is 100 square inches. A plane bisects three lateral edges of the pyramid. Find the area of the section made by the plane.

13. The altitude of a cone is 14 inches and the area of the base is 50 square inches. Find the area of a section parallel to the base and 2 inches from it.

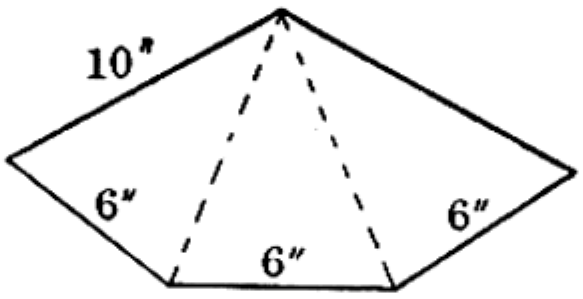
14. The area of a section of a cone parallel to the base is 50 square inches, and 6 inches from the base. If the altitude of the cone is 14 inches, find the area of the base.

15. The altitude of a cone is 24 inches. How far from the base must a plane be passed to form a parallel section whose area is half that of the base? one-third that of the base?

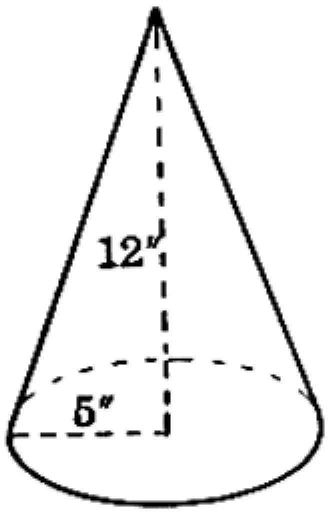
16. If the distance from the vertex to a section of a pyramid or a cone is doubled, by what number is the area of the section multiplied? if the distance is trebled? if it is multiplied by 10? by $\frac{1}{2}$? by n?

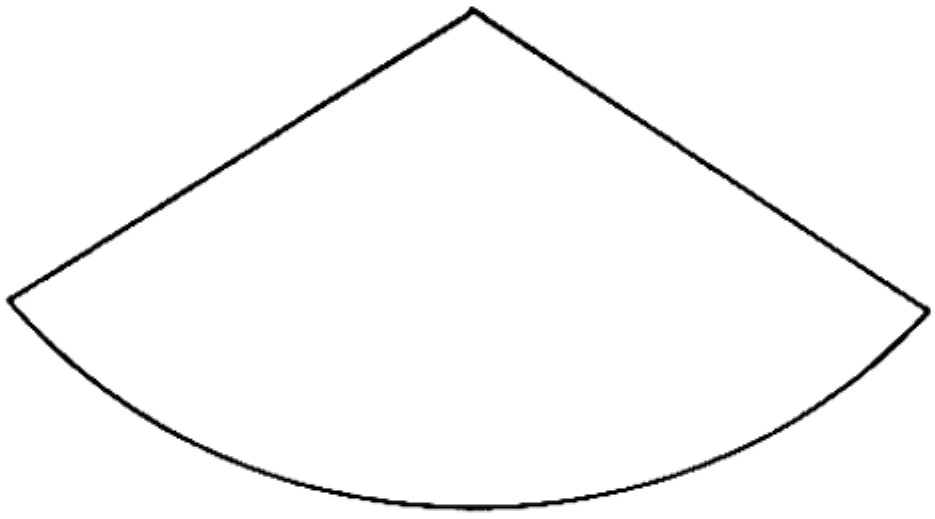
17. The areas of two parallel sections of a pyramid or a cone are 100 and 16 square inches. If they are 6 inches apart, how far from the vertex is the smaller section?

18. The pattern shown can be folded along the dotted lines and made into the lateral surface of a regular pyramid. Find the slant height, the lateral area, and the area of the base of the pyramid.



19. In the figure the sector is made by rolling out the lateral surface of the right circular cone. The altitude of the cone is 12 inches, and the radius of the base is 5 inches.





- **a.** Find the area of the sector.
- **b.** Find the number of degrees in the central angle of the sector.
- **c.** What would be the area of the sector if the given measurements were doubled? What would be the central angle of the sector?

20. A sector whose radius is 12 inches and whose central angle is 90° is rolled to form the lateral surface of a circular cone. Find the area of the base of the cone.

21. In a certain square pyramid the lateral faces are equilateral triangles. Find the inclination of the lateral edges to the plane of the base.

22. If two pyramids like that of the preceding problem have a common base, they form a regular octahedron (page 71).

- **a.** Draw a figure illustrating a regular octahedron.
- **b.** How many diagonals does your figure have?
- **c.** Prove that the diagonals are mutually perpendicular and bisect one another.

23. Find the total area of a regular octahedron if each edge is 10 inches.

24. Find the total area of a regular octahedron if each diagonal is $12\sqrt{2}$.

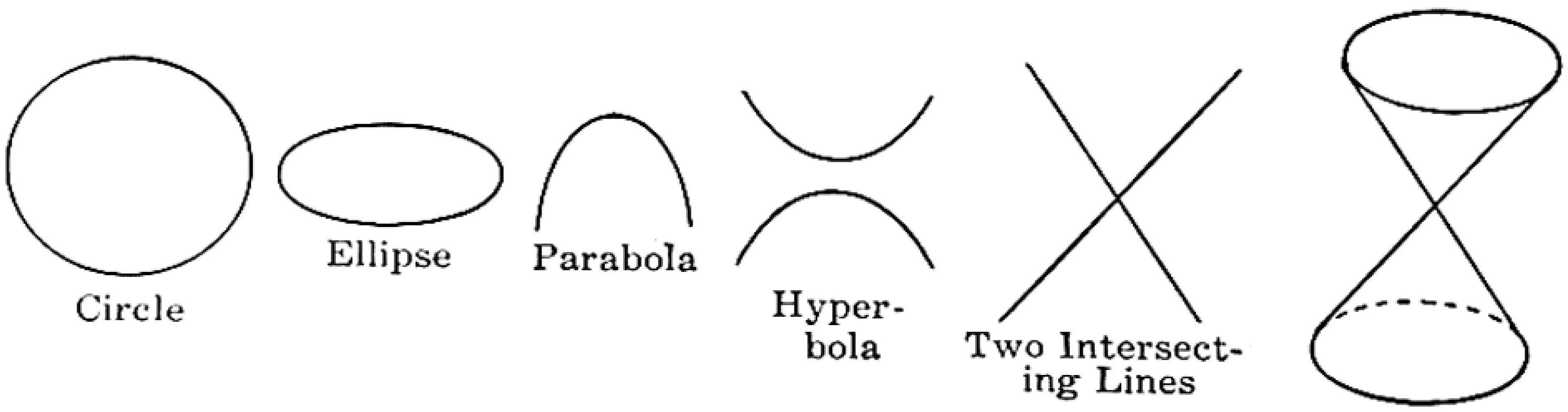
25. Prove that if three equal line segments bisect one another at right angles, their ends can be the vertices of a regular octahedron.

26. Prove that if one diagonal of a regular octahedron is m, the total surface is $m^2\sqrt{3}$.

27. In a certain square pyramid, the lateral faces are inclined 45° to the base. If the area of the base is 36 square inches, find the lateral edge of the pyramid.

28. In a regular triangular pyramid the lateral edges, each 18 inches, are inclined 60° to the base. Find the area of the base.

29. The five figures in the diagram are plane sections of the conical surface of the double right circular cone. You may recall from your algebra



that these are the only graphs that may result from plotting a second-degree equation in x and y.

Describe how the intersecting plane must be related to the vertex, the elements, the axis, or the base of the cone to produce each of these sections.

30. If the vertex part of a pyramid or a cone is cut away by a plane parallel to the base, the part that is left is called a *frustum* of the pyramid or the cone.

a. Prove that the lateral faces of a frustum of a pyramid are trapezoids.
b. Prove that the lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids.
c. In a frustum of a regular pyramid how is the part of the slant height that has not been cut away related to the trapezoid?
d. Find the lateral area of the frustum of a regular triangular pyramid if the perimeter of the upper base is 21 inches, of the lower base 33 inches, and the altitude of each trapezoid is 5 inches.

SUPERIOR WORK

S 1. In the diagram for the theorem in § 216 think of the base of the pyramid in terms of motion and answer the following.

a. If the base of a pyramid is translated, its area varies as —?—. If B is the variable area of the base and d is its variable distance from the vertex, the equation is $B = kd^2$.
b. If the base of a pyramid is 14 square inches when it is 6 inches from the vertex, what is its area when it is 9 inches from the vertex?

S 2. In the diagram for the theorem in § 216, V is a source of light that illuminates the plate $ABCD$. . . . If the plate is translated, the same amount of light will fall on the area $A'B'C'D'$ . . . because light travels in straight lines. The amount of light that falls on each unit of area is called the *intensity of illumination.*

a. If I is the intensity of illumination on $ABCD$ . . ., and I' is the intensity of illumination on $A'B'C'D'$ . . ., how is $\frac{I}{I'}$ related to the areas of the polygons?

b. Using the theorem in § 216, how is $\frac{I}{I'}$ related to the distances VH and VH'?

c. Complete the following statement: The intensity of illumination varies —?— of the distance from the source of light.

The statement in (c) exemplifies the *law of inverse squares.* It applies to action at a distance such as light, radiated heat, magnetism, radio waves, gravity, and so on. It may be written $I = \frac{k}{d^2}$, where I is the intensity of the action and d is the distance from the source.

S 3. Using the law of inverse squares of the preceding exercise, solve the following problem: If a magnet can lift a weight of 1 ounce at a distance of 3 inches, what weight can it lift at a distance of 2 inches?

S 4. a. A radio station 10 miles from your home is about to be moved to a point 40 miles from your home. Calling your present signal strength 1, what will be your new signal strength?

b. If at the same time that the station is moved, its power is increased from 5 kilowatts to 50 kilowatts, how will your new signal strength compare with the original?

S 5. An electric heater of 660 watts supplies a comfortable amount of heat at a distance of 6 feet. At what distance should an 1100-watt heater be placed to furnish the same amount of heat? *Help:* The intensity of the heat varies directly as the power P of the heater but inversely as the square of its distance d; that is, $I = k\frac{P}{d^2}$. Since I is to remain constant, $\frac{P}{d^2} = k'$.

S 6. Three edges of a cube meet in O. A plane cuts these edges in A, B, and C so that $OA = 9$ inches, $OB = 12$ inches, and $OC = 15$ inches. Find the distance from A to BC correct to three places.

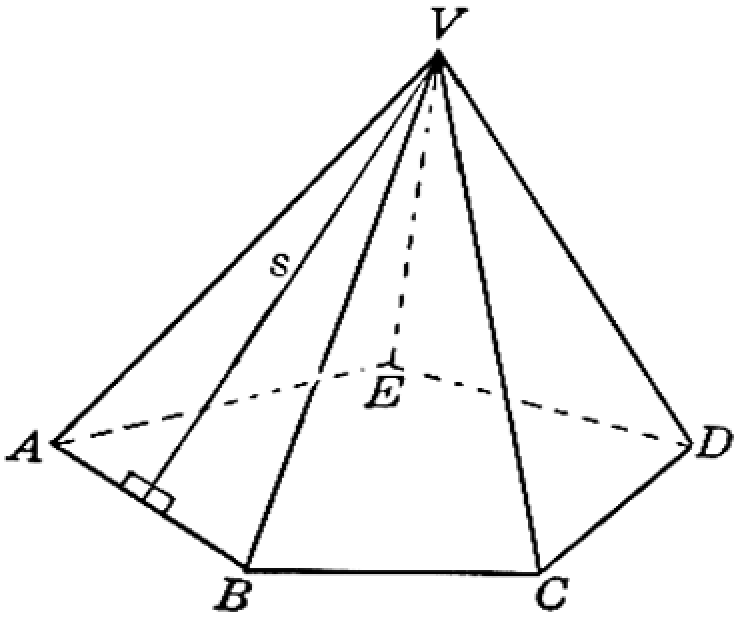
S 7. A section of a tetrahedron $ABCD$ containing CD and perpendicular to AB intersects the faces ABC and ABD in CE and ED respectively. If the bisector of the angle CED meets CD in F, prove that $CF:DF =$ area ABC:area ABD.

Measuring Pyramids and Cones

AREAS

To find the lateral area of a pyramid, we must find the areas of the triangles that make up the lateral surface. In the case of a regular pyramid the altitudes of these triangles from their common vertex is the same as the slant height. The sum of the areas of the triangles can therefore be broken up into two factors, one factor being half the slant height, and the other the perimeter of the base.

218. THEOREM. The lateral area of a regular pyramid equals half the product of the slant height and the perimeter of the base.



Given: L = lateral area of regular pyramid $V\text{–}ABCD\cdots$.
s = slant height.
$p = AB + BC + CD + \cdots$.

Prove: $L = \frac{1}{2}sp$.

Write out the proof.

Postulate § 213 permits us to apply this theorem to the circumscribed cone.

219. Corollary. The lateral area of a right circular cone equals half the product of the slant height and the circumference of the base.

$L = \frac{1}{2}sC$, where C is the circumference of the base.
Since $C = 2\pi r$, $L = \frac{1}{2}s \cdot 2\pi r = \pi rs$.
If T is the total area, $T = \pi r^2 + \pi rs = \pi r(r + s)$.

220. Corollary. If T is the total area of a right circular cone, r the radius of its base, and s its slant height, then $T = \pi r(r + s)$.

Frustums

221. Definitions. *A truncated pyramid or cone is the part of a pyramid or cone between the base and a section made by a plane not parallel to the base and cutting all the elements. The base of the pyramid or the cone and the section of the intersecting plane are called the bases of the truncated solid.*

222. Definitions. *A frustum of a pyramid or a cone is the part of a pyramid or a cone between the base and the section of a plane parallel to the base. The base of the pyramid or cone and the section* *of the intersecting plane are called the bases of the frustum; the rest of the surface is called the lateral surface. The altitude of a frustum is the perpendicular distance between the bases.*

223. Definition. *The faces of the lateral surface of a frustum of a pyramid are its lateral faces.*

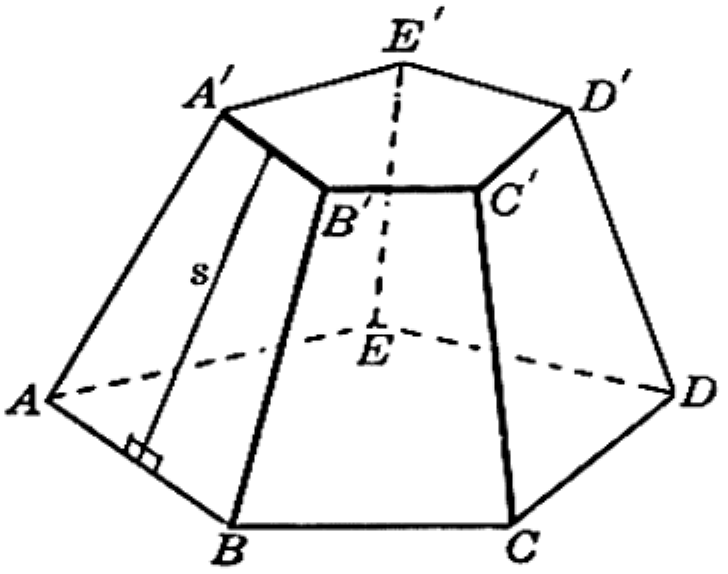
224. Corollary. The lateral faces of a frustum of a pyramid are trapezoids.

225. Corollary. The lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids.

226. Definitions. *The slant height of a frustum of a regular pyramid is the altitude of any one of its lateral faces. The slant height of a frustum of a right circular cone is the part of the slant height of the cone between the bases of the frustum.*

In a frustum of a regular pyramid, the area of each isosceles trapezoid equals half the product of the slant height and the sum of the bases. The sum of these areas will therefore have the common factor half the slant height, the other factor being the sum of the perimeters of the bases.

227. THEOREM. The lateral area of a frustum of a regular pyramid equals half the product of the slant height and the sum of the perimeters of the bases.



GIVEN: L = lateral area of frustum $A'D$ of regular pyramid.
s = slant height.
$p = AB + BC + CD + \cdots$.
$p' = A'B' + B'C' + C'D' + \cdots$.

PROVE: $L = \frac{1}{2}s(p + p')$.

Write out the proof.

Applying Postulate § 213, we have the corresponding proposition for the lateral area of a frustum of a right circular cone.

228. Corollary. The lateral area of a frustum of a right circular cone equals half the product of the slant height and the sum of the circumferences of the bases.

$$L = \tfrac{1}{2}s(C + C')$$

EXERCISES

1. Find the lateral area of a regular triangular pyramid if each basal side is 6 inches and the slant height is 5 inches.

2. Each basal side of a regular quadrangular pyramid is 10 inches and its altitude is 12 inches. Find the lateral area of the pyramid.

3. In the following problems s is the slant height of a frustum of a cone of revolution; r and r' are the radii of the bases. Find the lateral area in each case.

   a. $s = 12$ inches, $r = 6$ inches, $r' = 4$ inches.
   b. $s = 7$ feet, $r = 4.5$ feet, $r' = 3.8$ feet.
   c. $s = 8.5$ inches, $r = 4.2$ inches, $r' = 2.9$ inches.
   d. $s = 4\frac{1}{4}$ inches, $r = 7\frac{1}{2}$ inches, $r' = 4\frac{1}{8}$ inches.

4. The lateral edge of a regular hexagonal pyramid is 10 inches and each basal side is 12 inches. Find the lateral area.

5. The altitude of a regular hexagonal pyramid is 12 inches and each basal side is 5 inches. Find the lateral area.

6. The altitude of a frustum of a regular pyramid is 4 inches. Its bases are squares 10 inches and 4 inches on a side. Find the total area of the frustum.

7. In a regular hexagonal pyramid the altitude is 6 inches and each side of the base is 8 inches. Find the total area of the frustum formed by a plane 3 inches from the base.

8. Prove that the lateral area of a frustum of a regular pyramid or a cone of revolution equals the product of the slant height and the perimeter of a section midway between the bases.

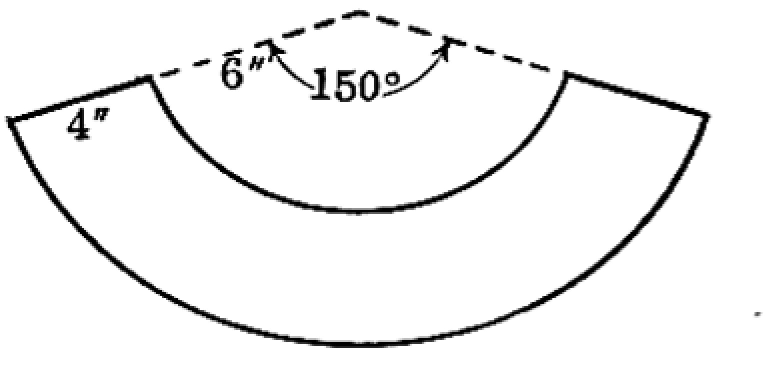
9. If s is the slant height of a frustum of a cone of revolution, r and r' are the radii of its bases, and L is its lateral area, prove that $L = \pi s(r + r')$.

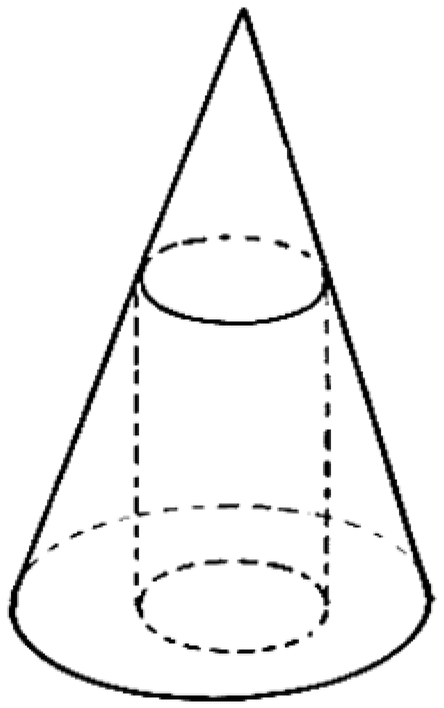
10. Two lines each 10 inches long bisect each other at right angles. Find the area generated by these lines as the figure is revolved about the bisector of the angle between the lines.

11. One end of a line segment 12 inches long is in a fixed plane. If the line rotates about this end always making an angle of 60° with the plane, find the area of the surface that it generates.

12. Solve the preceding problem if the line segment is bisected by the fixed plane and rotates about its midpoint.

13. The pattern in the figure is to be cut out along the solid lines and rolled up to form the lateral surface of a frustum of a cone of revolution. Using the measurements given, find the lateral area of the frustum.





14. In the figure a right circular cylinder fits into a right circular cone, the lower bases being coplanar and the axes collinear. The altitude of the cylinder is 6 inches and its lateral area is 48π square inches. Find the lateral area of the cone if its altitude is 9 inches.

15. The slant height of a regular quadrangular pyramid is 16 inches and makes an angle of 60° with the base. Find the total area of the pyramid.

16. The basal side of a regular triangular pyramid is 12 inches. The slant height forms an angle of 60° with the altitude. Find the total area of the pyramid.

17. The radius of the base of a cone of revolution is r and its altitude is h. Write a formula for the lateral area L in terms of r and h.

18. Find the total area of each of the five regular polyhedrons (page 71), assuming that each edge is $10\sqrt{3}$ inches. (You may use trigonometric functions for the regular dodecahedron.)

19. A right circular cylinder and a right circular cone have equal bases, but the altitude of the cylinder equals the slant height of the cone. How do their lateral areas compare?

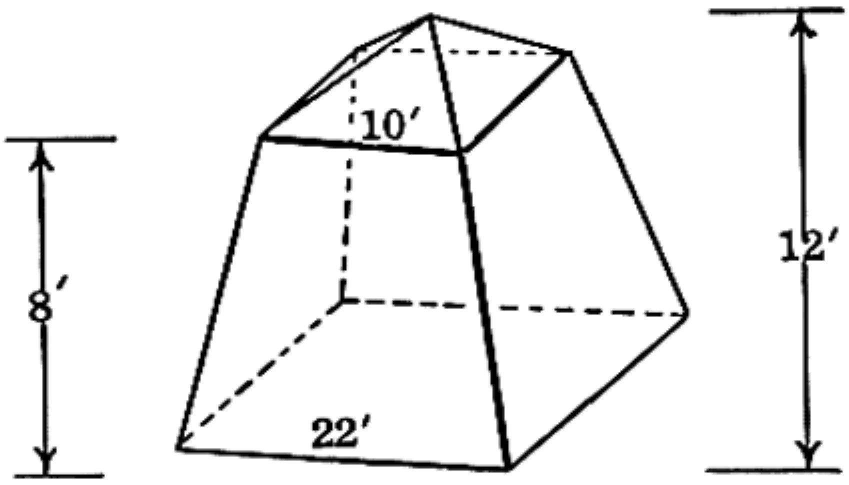
20. One lateral edge of a rectangular pyramid is perpendicular to the base. The altitude of the pyramid is 7 inches and the dimensions of the base are 8 inches and 9 inches. Find the total area of the pyramid.

21. Find the total area of the frustum of the pyramid in Exercise 20 made by a plane 4 inches from the base.

22. Prove that if a plane section of a frustum of a cone contains an element, it is a trapezoid. (Use § 214.)

23. A garage is 24 feet by 24 feet. The roof is in the form of a regular pyramid 5 feet high. Find the area of the roof.

24. The figure shows a tent made in the form of a frustum of a regular quadrangular pyramid with another regular quadrangular pyramid superimposed. Using the measurements in the diagram, find the number of square feet of canvas needed. Allow 20 per cent for cutting and joining.



25. The altitude of a regular tetrahedron is 12 inches. Find the total area.

SUPERIOR WORK

S 1. The total area of a regular tetrahedron is $25\sqrt{3}$ inches. Find the lateral area of the circumscribed cone.

S 2. In a frustum of a triangular pyramid lines are drawn from the vertices of one base to the midpoints of the opposite sides of the other base. Prove that these lines are concurrent.

S 3. The altitude of a regular pentagonal pyramid is 9 inches and each lateral edge is 15 inches. Find the lateral area of the pyramid.

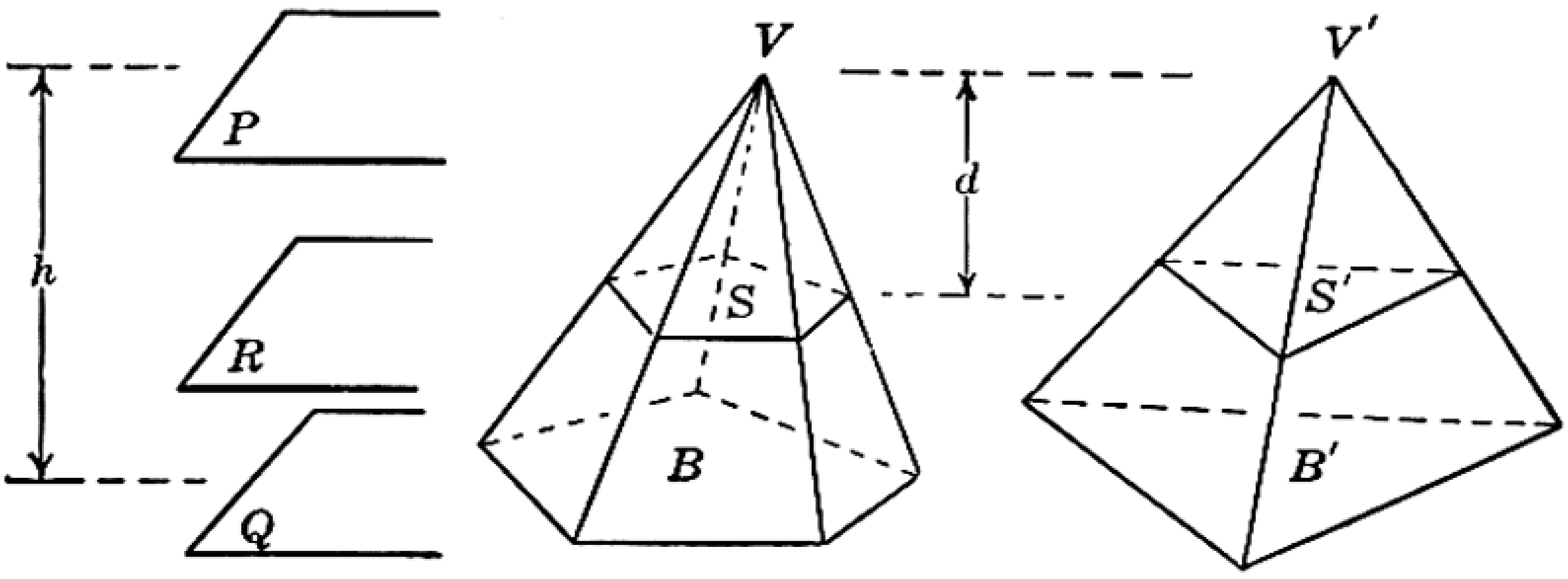
S 4. The lateral area of a right circular cone is 100 square inches and its altitude is 5 inches. Find the radius of the base.

S 5. The lower base of a frustum of a right circular cone is also the base of a cone whose vertex is the center of the upper base; the upper base of the frustum is also the base of a cone whose vertex is the center of the lower base. If the diameters of the bases are 10 inches and 40 inches, find the area of the circle in which the cones intersect.

VOLUMES

229. To find the volume of a pyramid, we shall compare it with a prism. We shall prove that a triangular prism can be cut up into three pyramids that have equal bases and equal altitudes. Therefore we first prove that such pyramids have equal volumes by showing that they are Cavalieri solids.

230. THEOREM. If two pyramids have equal bases and equal altitudes they are equal.



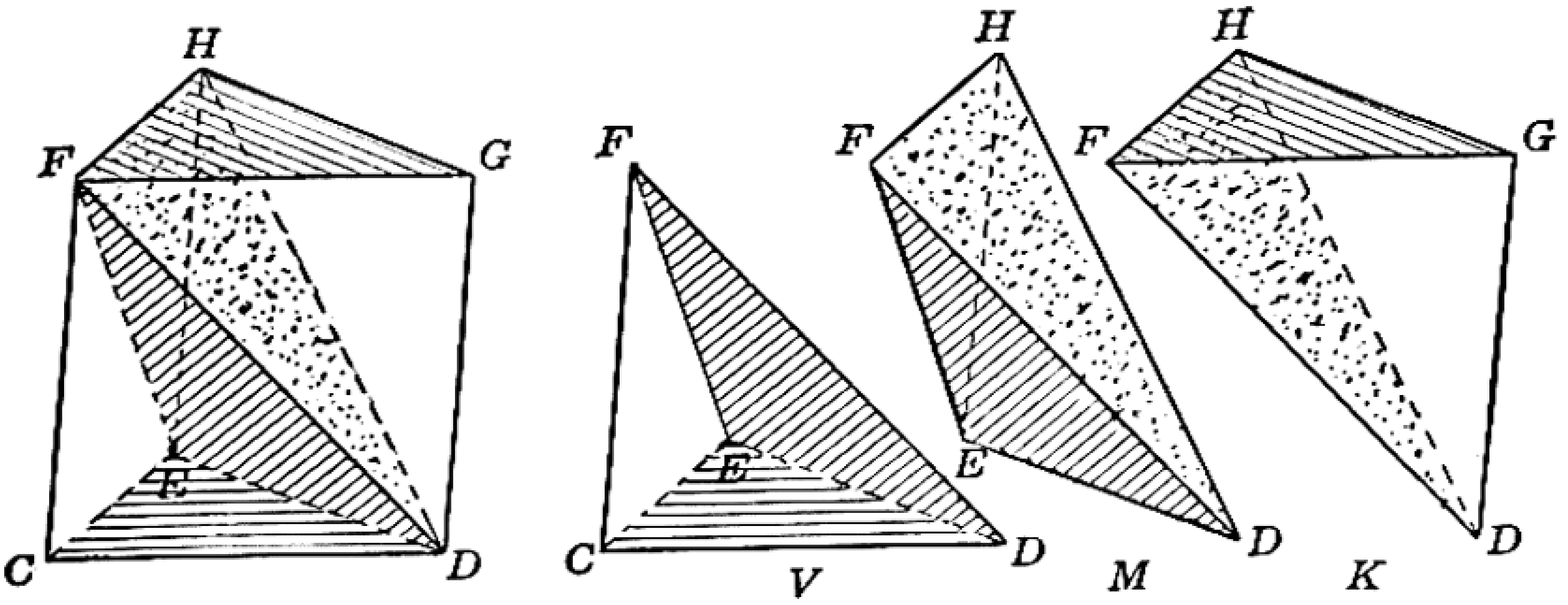
Given: Pyramids V and V' with equal bases B and B' and equal altitudes.

Prove: $V = V'$.

CONS.: Since their altitudes are equal, place the pyramids with their bases in plane Q and their vertices in plane P, which is parallel to Q and a distance h from Q.
Draw *any* third plane R parallel to P, cutting V in S and V' in S'.
Let $d =$ the distance between P and R.

PROOF: *Statements*	*Reasons*
1. $\frac{S}{B} = \frac{d^2}{h^2}, \frac{S'}{B'} = \frac{d^2}{h^2}.$	1. Section of pyramid ‖ to base is to base as sq. of its distance from vertex is to sq. of alt. (§ 216)
2. $\frac{S}{B} = \frac{S'}{B'}.$	2. Subs. Ax.
3. $B = B'.$	3. Given.
4. $S = S'.$	4. Mult. Ax.
5. $V = V'.$	5. Cavalieri's Proposition. (§ 176)

231. To understand how a prism can be divided into three equal pyramids, perform the following experiment.

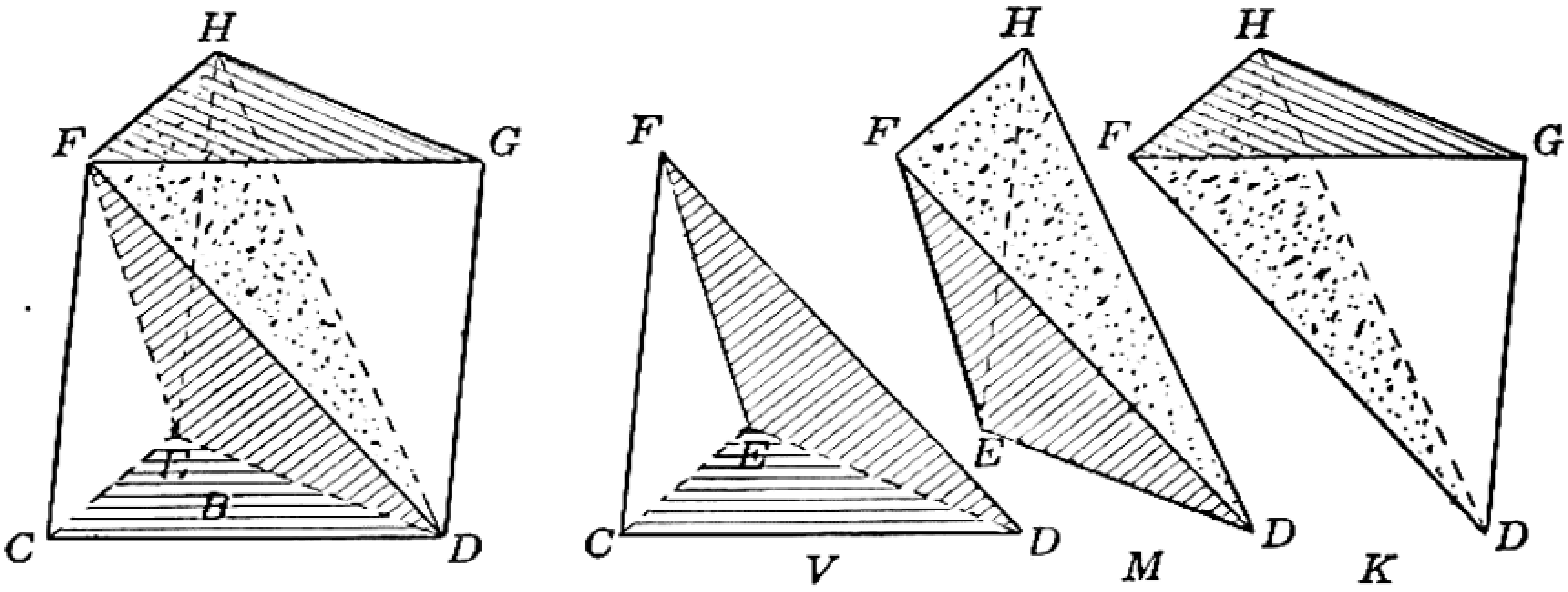


From a potato cut out a triangular prism like the one shown in the first figure. Cut this prism along the plane FDE. This will divide the prism into the triangular pyramid $F\text{–}CDE$, which we will call V, and the quadrangular pyramid $F\text{–}DGHE$. Cut this into two triangular pyramids $F\text{–}DEH$ and $F\text{–}DGH$, which we will call M and K respectively.

a. Your two end pyramids V and K have equal bases. Name them. Describe the altitudes to these bases. How do they compare? Why? Do you see that there are two ways to answer these questions? Explain.

b. Pyramid V and the middle pyramid M also have equal bases. Name them. Describe the altitudes to these bases. How do they compare?

232. THEOREM. The volume of a triangular pyramid equals one-third the product of its base and altitude.



GIVEN: V = volume of triangular pyramid $F\text{–}CDE$.
B = area of its base CDE.
h = altitude from vertex F.

PROVE: $V = \frac{1}{3}Bh$.

CONS.: With CDE as a base and FC as a lateral edge, construct the triangular prism FD. Pass a plane through FD and FH intersecting face GE in HD.
V = given pyramid $F\text{–}CDE$ or $D\text{–}CEF$.
Let K = pyramid $D\text{–}FGH$, and M = pyramid $D\text{–}EFH$.

PROOF:

Statements	*Reasons*
In pyramids V and K:	
1. Base CDE = base FGH.	1. Bases of a prism are =.
2. Alt. from F = alt. from D.	2. ∥ planes are everywhere equidistant. (§ 56)
3. $V = K$.	3. If 2 pyramids have = bases and = alts. they are =. (§ 230)
In pyramids V and M:	
4. Base CEF = base EFH.	4. Diagonal of ▱ divides it into $\cong$ △s. (§ 506)
5. Alt. from D = alt. from D.	5. Identity.
6. $V = M$.	6. Same as 3.
7. $V + M + K$ = prism FD.	7. Identity.
8. Prism $FD = Bh$.	8. Volume of prism = base × alt. (§ 180)
9. $V + V + V = Bh$. or $3V = Bh$.	9. Subs. Ax.
10. $V = \frac{1}{3}Bh$.	10. Div. Ax.

233. In plane geometry you learned that any polygon may be transformed into a triangle. Any given pyramid can therefore be transformed into another pyramid having the same altitude and a triangular base equal to the given base. Our theorem is therefore true for any pyramid. Furthermore, Postulate § 213 permits us to apply it to a cone. We therefore have the following general proposition.

234. Corollary. The volume of any pyramid or circular cone equals one-third the product of its base and altitude.

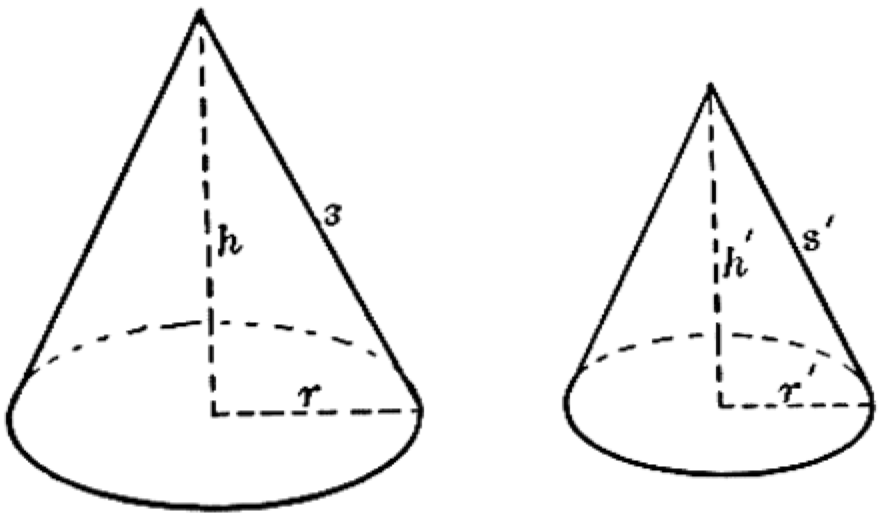
Note: It can be proved that this corollary is true also for a cone whose base is not a circle.

Similar Cones of Revolution

We shall now compare two similar cones of revolution. Compare the following definition and theorem with § 183 and § 184.

235. Definition. *Similar cones of revolution are cones that are generated by revolving similar right triangles about corresponding legs as axes.*

236. THEOREM. The lateral areas or the total areas of two similar cones of revolution are proportional to the squares of their altitudes, or to the squares of their slant heights, or to the squares of the radii of their bases; their volumes are proportional to the cubes of their altitudes, or to the cubes of their slant heights, or to the cubes of the radii of their bases.



GIVEN: In the $\sim$ cones of revolution,

L and L' the lateral areas.
T and T' the total areas.
V and V' the volumes.
h and h' the altitudes.
s and s' the slant heights.
r and r' the radii of the bases.

PROVE: $\frac{L}{L'} = \frac{T}{T'} = \frac{h^2}{h'^2} = \frac{s^2}{s'^2} = \frac{r^2}{r'^2}.$

$\frac{V}{V'} = \frac{h^3}{h'^3} = \frac{s^3}{s'^3} = \frac{r^3}{r'^3}.$

PROOF: *Statements* Supply the reasons.

1. $\frac{r}{r'} = \frac{s}{s'} = \frac{h}{h'}$.

2. $\frac{r+s}{r'+s'} = \frac{r}{r'} = \frac{s}{s'} = \frac{h}{h'}$.

3. $L = \pi rs$.
   $L' = \pi r's'$.

4. $\frac{L}{L'} = \frac{rs}{r's'} = \frac{r}{r'}\cdot\frac{s}{s'} = \frac{r}{r'}\cdot\frac{r}{r'} = \frac{r^2}{r'^2} = \frac{s^2}{s'^2} = \frac{h^2}{h'^2}$.

5. $T = \pi r(r+s)$.
   $T' = \pi r'(r'+s')$.

6. $\frac{T}{T'} = \frac{r(r+s)}{r'(r'+s')} = \frac{r}{r'}\cdot\frac{r+s}{r'+s'} = \frac{r}{r'}\cdot\frac{r}{r'} = \frac{r^2}{r'^2} = \frac{s^2}{s'^2} = \frac{h^2}{h'^2}$.

7. $V = \frac{1}{3}\pi r^2 h$.
   $V' = \frac{1}{3}\pi r'^2 h'$.

8. $\frac{V}{V'} = \frac{r^2h}{r'^2h'} = \frac{r^2}{r'^2}\cdot\frac{h}{h'} = \frac{r^2}{r'^2}\cdot\frac{r}{r'} = \frac{r^3}{r'^3} = \frac{s^3}{s'^3} = \frac{h^3}{h'^3}$.

Similar Polyhedrons

237. Definition. *Similar polyhedrons are polyhedrons in which the faces are similar each to each and similarly placed, and their corresponding polyhedral angles are equal.*

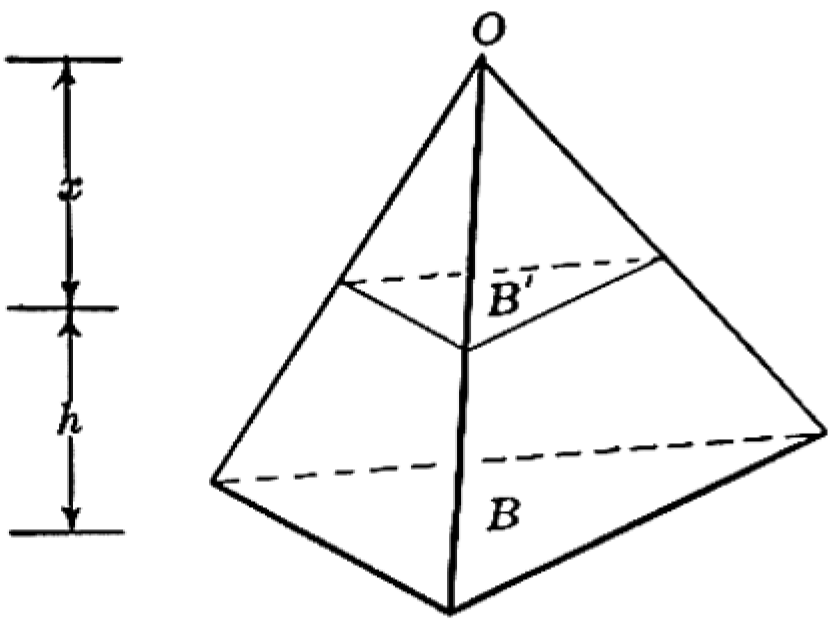
238. It can be proved that the theorems about similar cylinders of revolution and similar cones of revolution are also true for similar polyhedrons. We shall state this without proof.

In two similar polyhedrons corresponding lines are proportional to corresponding lines; corresponding areas are proportional to the squares of corresponding lines; corresponding volumes are proportional to the cubes of corresponding lines.

The Volume of a Frustum

239. PROBLEM. To find the volume of a frustum of a pyramid.

Analysis: The given frustum is the difference between the original pyramid and the pyramid that has been cut away to form the frustum. This suggests that we rebuild the original pyramid. To find the altitudes of the pyramids, we make use of the theorem (§ 216) that the given bases are proportional to the squares of these altitudes. The problem is therefore an algebraic exercise.



GIVEN: Frustum of pyramid with bases B and B' and altitude h.

REQ.: To find V, the volume of the frustum.

CONS.: Prolong the lateral edges to meet in O. (Def. of frustum).

Let x = alt. of the smaller pyramid.

SOLUTION:

1. $$\frac{B'}{B} = \frac{x^2}{(x+h)^2}.$$

Let $B = q^2$
and $B' = q'^2$.

2. $$\frac{q'^2}{q^2} = \frac{x^2}{(x+h)^2}.$$

$$\frac{q'}{q} = \frac{x}{x+h}.$$

Solving for x, $x = \dfrac{q'h}{q-q'}$.

3. $$\begin{aligned} V &= \tfrac{1}{3}B(x+h) - \tfrac{1}{3}B'x, \\ &= \tfrac{1}{3}q^2(x+h) - \tfrac{1}{3}q'^2x, \\ &= \tfrac{1}{3}x(q^2 - q'^2) + \tfrac{1}{3}q^2h. \end{aligned}$$

Substituting the value of x,

$$\begin{aligned} V &= \frac{1}{3}\cdot\frac{q'h}{q-q'}(q^2 - q'^2) + \frac{1}{3}q^2h, \\ &= \frac{1}{3}\cdot\frac{q'h}{q-q'}(q+q')(q-q') + \frac{1}{3}q^2h = \frac{1}{3}q'h(q+q') + \frac{1}{3}q^2h, \\ &= \tfrac{1}{3}h(q^2 + q'^2 + q\cdot q'). \end{aligned}$$

Since $q = \sqrt{B}$, and $q' = \sqrt{B}'$,

$$V = \tfrac{1}{3}h(B + B' + \sqrt{BB'}).$$

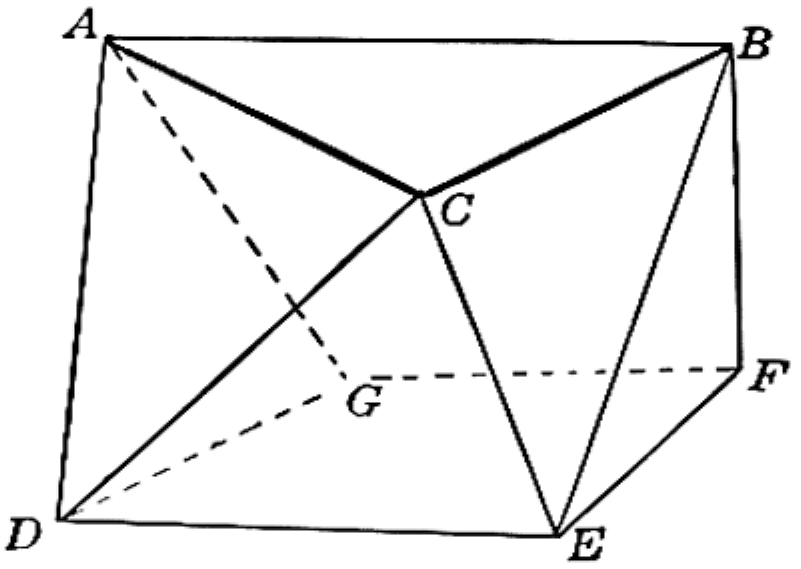
Using Postulate § 213, this formula may also be applied to a frustum of a cone.

240. THEOREM. If the bases of a frustum of a pyramid or a cone are B and B', its altitude is h, and its volume is V, then $V = \frac{1}{3}h(B + B' + \sqrt{BB'})$.

The Prismatoid (Optional)

241. Definition. *A prismatoid is a polyhedron whose vertices all lie in two parallel planes. The faces that lie in the parallel planes are called the bases of the prismatoid, and the perpendicular distance between the bases is the altitude.*

A prismatoid is a more general polyhedron than a prism or a pyramid, and it includes both as special cases. The following proposition for finding the volume of a prismatoid can be proved.



Plane $ABC \parallel$ plane $DEFG$
Polyhedron AF is a Prismatoid

242. THEOREM. The volume of a prismatoid equals one-sixth the altitude times the sum of the areas of the bases and four times the area of a plane section midway between the bases.

If the bases are B and B', the midsection M, and the altitude h, then the volume $V = \frac{1}{6}h(B + 4M + B')$.

EXERCISES

1. Each side of the base of a square pyramid is 3 feet and its altitude is 5 feet. Find the volume of the pyramid.

2. The base of a pyramid is a rectangle 5 feet by 3 feet. The altitude is 4 feet 4 inches. Find the volume of the pyramid.

3. The altitude of a square pyramid is 9 inches and the volume is $\frac{1}{9}$ cubic foot. Find each side of the base.

4. Find the volume of a regular quadrangular pyramid whose slant height is 15 inches and whose basal side is 6 inches.

5. A side of the base of a regular triangular pyramid is 6 inches and the slant height is 15 inches. Find the volume of the pyramid.

6. Find the volume of a cone in which the radius of the base is 4 inches and the altitude is 7 inches.

7. Find the volume of a cone whose altitude is 12 inches and whose slant height is 13 inches.

8. The volume of a cone equals 36π cubic inches and its altitude is 4 inches. Find the lateral area of the cone if it is a cone of revolution.

9. If the base of a pyramid is a parallelogram, prove that two planes, each through two opposite lateral edges of the pyramid, divide the figure into four equal pyramids.

10. Three cones have equal altitudes. The radii of two cones are 8 inches and 6 inches respectively. Find the radius of a third cone if it is equal to the sum of the first two cones.

11. Find the volume of a regular hexagonal pyramid whose lateral edge is 18 inches long and makes an angle of 30° with the altitude.

12. The slant heights of two similar cones of revolution are 5 inches and 18 inches. If the volume of the smaller cone is 200 cubic inches, find the volume of the larger cone.

13. The volumes of two similar cones of revolution are $1\frac{1}{2}$ cubic feet and 12 cubic feet. If the altitude of the larger cone is 14 inches, find the altitude of the smaller cone.

14. The volumes of two similar cones of revolution are 24 and 81 cubic inches. If the lateral area of the smaller cone is 8 square inches, find the lateral area of the larger cone.

15. **a.** An equilateral triangle revolves about one side as an axis. Find the area of the solid generated if one side is 10.

b. Solve (a) if one side is s.

16. The axis of a circular cone is 10 inches and makes an angle of 45° with the altitude of the cone. If the radius of the base is 6 inches, find the volume of the cone.

17. The altitude of a cone of revolution is increased 10 per cent. If the new cone is similar to the original cone, by what per cent has the volume increased?

18. The axis section of a right circular cone is an equilateral triangle. Find the volume of the cone in terms of its altitude h.

19. A gallon container is in the form of a right circular cone. At what distances from the vertex along the slant height should the quart marks be placed?

20. A metal cone 6 inches high weighs 6 pounds. It is cut into three parts by planes parallel to the base, the cuts being 2 inches and 4 inches from the vertex. Find the weight of each part.

21. Find the volume of a regular tetrahedron whose edge is 12 inches.

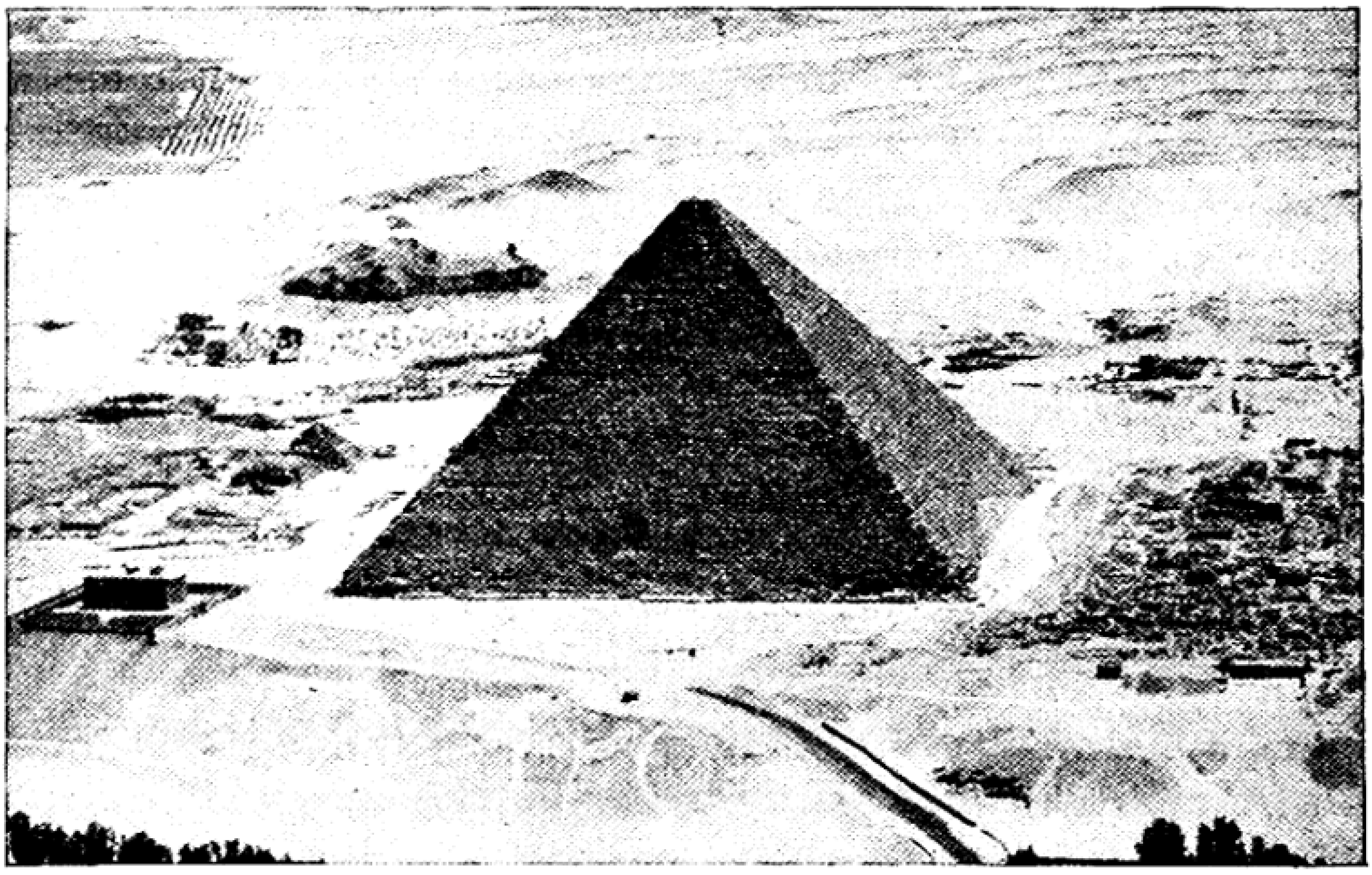
22. Find the volume of the cone inscribed in the tetrahedron in Exercise 21.

23. Find the volume of the cone circumscribed about the tetrahedron of Exercise 21.

24. Prove that the sum of the distances from a point within a regular tetrahedron to the four faces equals the altitude of the tetrahedron. *Hint:* Pass planes through the given point and the edges of the tetrahedron, dividing the figure into four pyramids. Find the volume of each pyramid.

What is the analogous proposition in plane geometry?

25. The Great Pyramid of Egypt was built by the tyrant Pharaoh Cheops about four thousand years ago with slave labor as a tomb for himself. When built it was 481 feet high and 756 feet square at the base. It is now 450 feet high and 746 feet square at the base. If one cubic yard of the stone weighs 2 tons, how many tons have weathered away?



26. The area of the base of a cone is 18 square inches and the altitude is 15 inches. It is cut by a plane parallel to the base and 10 inches from it.

a. Find the area of the cross section.
b. Find the volume of the frustum.

27. The altitude of a frustum of a circular cone is 6 inches. The radii of the bases are 3 inches and 4 inches.

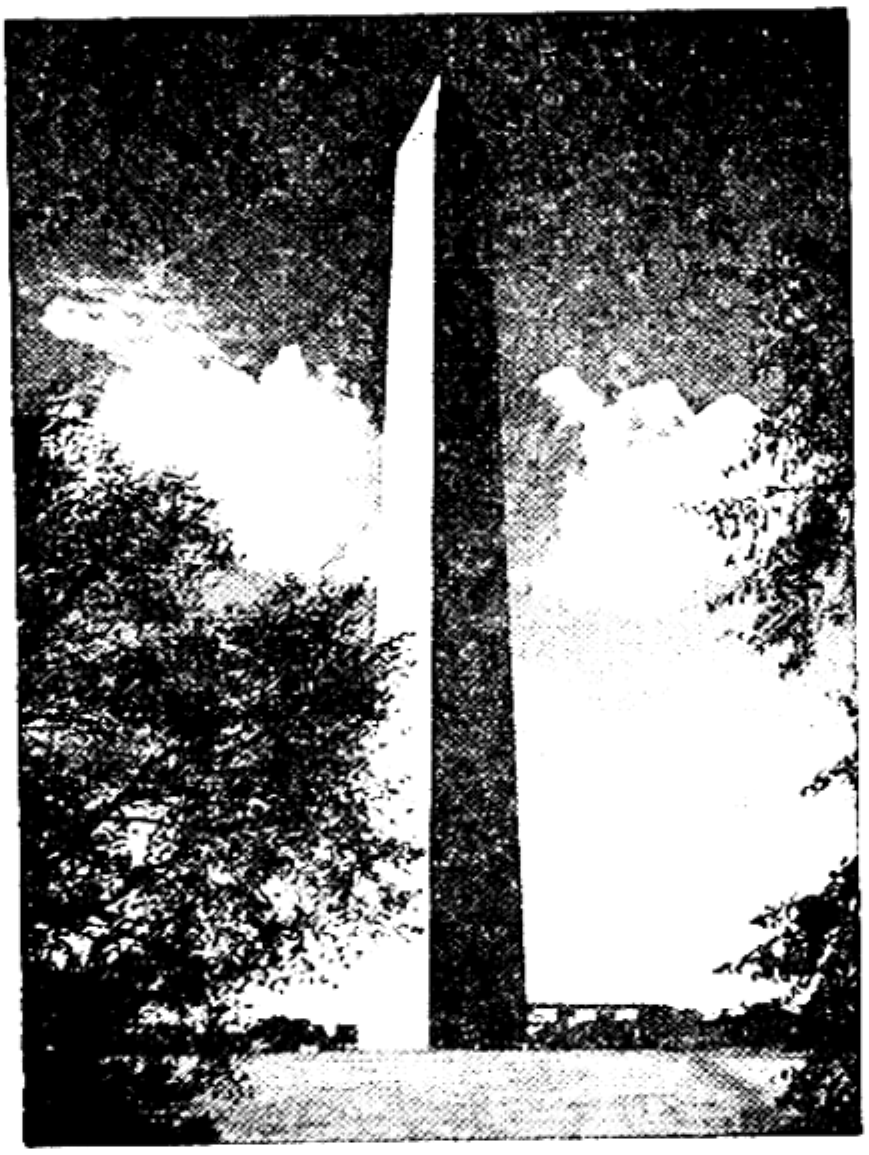
a. Find the volume of the frustum by first finding the altitude of the original cone and then finding the volumes of the two cones.
b. Check your result in (a) by substituting in the formula of § 240.

28. The altitude of a frustum of a triangular pyramid is 6 inches. The sides of one base are 3, 4, and 5 inches; the sides of the other base are 9, 12, and 15 inches. Find the volume of the frustum.

29. The radii of the bases of a frustum of a right circular cone are r and r' and its altitude is h. Prove that the volume $V = \frac{1}{3}\pi h(r^2 + r'^2 + rr')$.

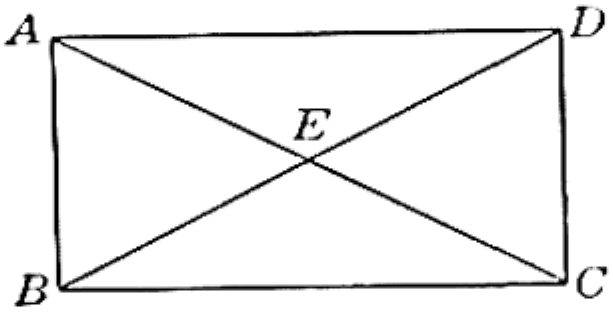
30. A watering trough is in the form of a frustum of a rectangular pyramid. The lower base is 30 inches by 12 inches, the upper base is 40 inches by 16 inches, and it is 9 inches deep. How many gallons can it hold? (1 gal. = 231 cu. in.)

SUPERIOR WORK



S 1. The Washington Monument in Washington, D.C., is in the form of a frustum of a square pyramid with a pyramid superimposed. The frustum is 500 feet high, 55 feet square at the base, and 34 feet square at the top. The pyramid surmounting the shaft is 55 feet high.

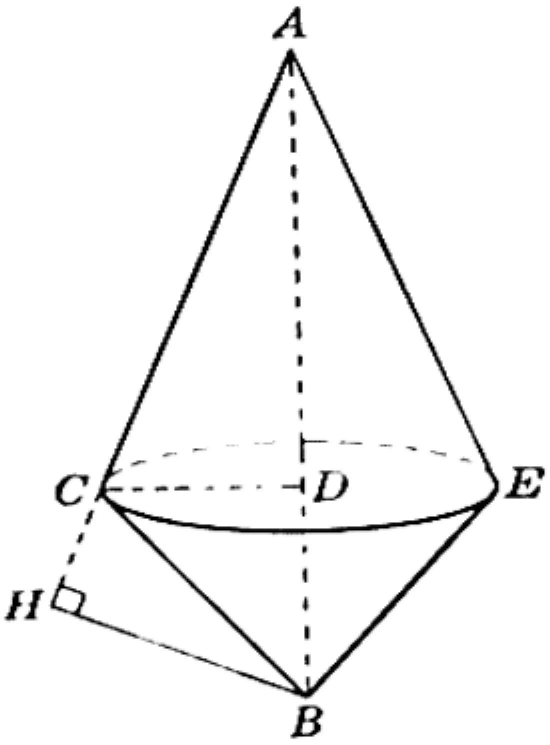
- **a.** Find the area of the outside lateral surface of the monument.
- **b.** Find the volume of the monument.
- **c.** The walls of the frustum are 15 feet thick at the bottom and 18 inches thick at the top. Find the number of cubic feet in the walls.



S 2. Rectangle $ABCD$ revolves about AB as an axis. Find the ratio of the volumes generated by triangles ABE and CDE.

S 3. Prove the following proposition.

THEOREM. The volume generated by a triangle revolving about one side as an axis equals one-third the altitude to one of the other sides times the area generated by that side.



GIVEN: V = volume generated by revolving $\triangle ABC$ about AB.
BH the altitude to AC.
L = area generated by AC.

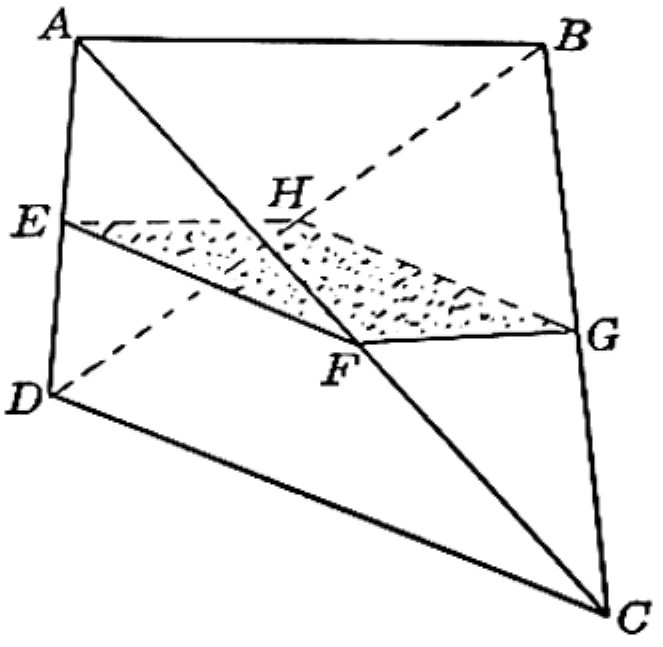
PROVE: $V = \frac{1}{3}BH \cdot L$.

Help:

$$\begin{aligned} V &= \text{cone } ACE + \text{cone } BCE \\ &= \tfrac{1}{3}\pi \cdot CD^2 \cdot AD + \tfrac{1}{3}\pi \cdot CD^2 \cdot DB \\ &= \tfrac{1}{3}\pi \cdot CD^2(AD + DB) \\ &= \tfrac{1}{3}\pi \cdot CD^2 \cdot AB. \\ L &= \pi \cdot CD \cdot AC. \\ \frac{AB}{AC} &= \frac{BH}{CD}. \end{aligned}$$

S 4. Find the volumes of the following solids by considering them as special prismatoids and using the formula in § 242: (*a*) a prism, (*b*) a pyramid, (*c*) a frustum of a pyramid.

S 5. Prove that the volume of a tetrahedron equals one-sixth the product of the shortest distance between two opposite edges and the area of a parallelogram whose sides are equal and parallel to these edges. *Hint:* E, F, G, and H are the midpoints of four lateral edges of the given tetrahedron. Consider the tetrahedron as a prismatoid whose "bases" are AB and DC.



SUMMARY

243. Pyramids and cones are formed in part by ruled surfaces that are generated by rotating a line. Our study of these figures was in many ways similar to the study of prisms and cylinders in the preceding chapter. Here we assumed that any proposition about a pyramid with a regular base which does not depend on the number of lateral faces may be applied to a cone.

Cavalieri's proposition enabled us to show that pyramids with equal bases and equal altitudes have equal volumes. By dissecting a triangular prism into three such pyramids, we found the volume of the pyramid.

We proved the following important theorems.

Every section of a pyramid or a cone made by a plane passing through the vertex is a triangle.

If a pyramid or a cone is cut by a plane parallel to the base, the lateral edges are divided proportionally; the section is a polygon (or a circle) similar to the base; the area of the section is to the area of the base as the square of its distance from the vertex is to the square of the altitude.

The lateral area of a regular pyramid or a right circular cone equals half the product of the slant height and the perimeter of the base.

The lateral area of a frustum of a regular pyramid or a right circular cone equals half the product of the slant height and the sum of the perimeters of the bases.

The volume of a pyramid or a cone equals one-third the product of its base and altitude.

In two similar cones of revolution the lateral areas or the total areas have the same ratio as the squares of the altitudes, or as the squares of the slant heights, or as the squares of the radii of the bases; but the volumes have the same ratio as the cubes of the altitudes, or as the cubes of the slant heights, or as the cubes of the radii of the bases.

The volume of a frustum of a pyramid or a cone is given by the formula $V = \frac{1}{3}h(B + B' + \sqrt{BB'})$, where h is the altitude and B and B' are the bases.

RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, go back to the section indicated.

- pyramidal surface § 188
  - nappes § 189
- conical surface § 188
  - nappes § 189
- closed pyramidal surface § 190
- closed conical surface § 190
- pyramid § 191
  - vertex § 192
  - base § 192
  - lateral surface § 192
  - lateral faces § 193
  - elements § 192
  - lateral edge § 194
  - altitude § 195
- cone § 191
  - vertex § 192
  - base § 192
  - lateral surface § 192
  - elements § 192
  - altitude § 195
- triangular pyramid § 196
- quadrangular pyramid, etc. § 196
- circular cone § 197
  - axis § 197
- regular pyramid § 198
  - axis § 198
  - slant height § 199
- oblique pyramid § 198
- right circular cone § 198
  - slant height § 199
- oblique cone § 198
- cone of revolution § 205
- plane tangent to cone § 206
- pyramid circumscribed about cone § 207
- pyramid inscribed in cone § 209
- equilateral cone § 215
- truncated pyramid § 221
  - bases § 221
- frustum of pyramid § 222
  - bases § 222
  - lateral surface § 222
  - lateral faces § 223
  - altitude § 222
- frustum of cone § 222
  - bases § 222
  - lateral surface § 222
  - altitude § 222
- slant height of frustum of regular pyramid § 226
- slant height of frustum of right circular cone § 226
- similar cones of revolution § 235
- similar polyhedrons § 237

REVIEW EXERCISES

1. One side of a quadrilateral which is 12 inches long is perpendicular to the two adjoining sides which are 9 inches and 14 inches. If the quadrilateral is rotated about the 12-inch side, find the area and the volume generated.

2. The altitude of a regular hexagonal pyramid is 15 inches and a side of its base is 6 inches. Find the total area and the volume of the pyramid.

3. The base of a certain pyramid is a triangle with sides 3, 4, and 5 inches. The altitude is 10 inches. Find the area of a section made by a plane parallel to the base and 7 inches from it.

4. The base of a pyramid is also one face of a cube, and the vertex of the pyramid is the center of the opposite face of the cube. Find the ratio of the volume of the pyramid to the volume of the cube; also the ratio of the total areas.

5. A plane is drawn through two face diagonals of a cube from the same vertex. What part of the volume of the cube is the volume of the pyramid thus formed?

6. A parchment lamp shade is to have an upper diameter of 14 inches and a lower diameter of 18 inches. The height is to be 1 foot. Allowing 20 per cent for construction, how many square yards of parchment are needed for 1000 shades?

7. A plane bisects three concurrent edges of a parallelepiped. Compare the volume of the resulting pyramid with the volume of the parallelepiped.

8. The total areas of two similar cones of revolution are 20 and 45 square inches respectively. If the volume of the larger cone is 54 cubic inches, find the volume of the smaller cone.

9. An isosceles right triangle with each leg 4 inches is rotated about the hypotenuse. Find the area and the volume generated.

10. A right triangle with legs 6 and 8 is rotated about the hypotenuse. Find the area and the volume generated.

11. Two solids lie between two parallel planes. A translated third plane cuts sections from these solids that are always in the ratio of 2:3. What is the ratio of the volumes of the solids?

12. One face of a tetrahedron is an equilateral triangle 5 inches on a side. The other three edges are 12, 13, and 13 inches. Find the volume of the tetrahedron.

13. A square pyramid 14 inches high, having a side of its base 4 inches, is melted and recast into a cone 10 inches high. Find the radius of the base.

14. A regular hexagon whose side is 12 inches revolves about one of its longer diagonals. Find the area and the volume generated.

15. The axis of a circular cone is 18 inches and it makes an angle of 60° with the base. Find the volume of the cone if the radius of the base is 8 inches.

16. A smokestack is 75 feet high. The lower base is a regular hexagon 8 feet on each side and the upper base is a regular hexagon 6 feet on each side. The inner flue is a cylinder 4 feet in diameter. Find the volume of the material in the smokestack.

17. a. What is the locus of a point which is equidistant from a given plane and a given perpendicular to the plane?
    b. What is the locus of a point which is twice as far from the line in (a) as from the plane?

18. The centers of the faces of a cube are the vertices of another polyhedron. Compare the area and the volume of this polyhedron with the area and the volume of the cube.

19. The diagonals of an octahedron are 12, 7, and 11 inches and they bisect one another at right angles. Find the volume of the octahedron.

20. The total area of a cone of revolution is increased by 44 per cent. If the new cone is similar to the original cone, by what per cent has the volume increased?

21. A rectangle is divided into two triangles by one of its diagonals. If the rectangle is revolved about one of its sides, what is the ratio of the two solids generated by the triangles?

22. An inverted cone 12 inches in diameter and $10\frac{2}{3}$ inches deep catches 1.5 inches of rainfall. How deep is the water in the cone?

23. The midpoint of the longer side of a rectangle is joined to the opposite vertices, dividing the rectangle into three triangles. Find the area and the volume generated by each triangle as the rectangle revolves about its shorter side. The dimensions of the rectangle are 6 inches by 4 inches.

24. The base of a pyramid is a trapezoid. Two planes are drawn, each plane through two opposite lateral edges, dividing the figure into four pyramids. Prove that two of these pyramids are equal.

25. Prove: If the diagonals of the base of a quadrangular pyramid are perpendicular to each other, the volume of the pyramid equals one-sixth the product of these diagonals and the altitude of the pyramid.

26. A cube $\sqrt{2}$ inches on a side is to be recast into a regular octahedron. How long will the edge of the octahedron be?

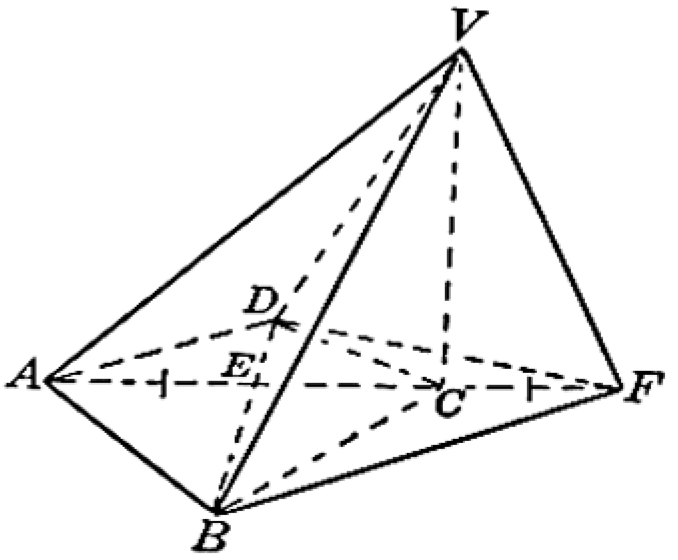
27. A 270° sector of a circle is rolled up to form the lateral surface of a cone of revolution. If the radius of the sector is 6 inches, find the volume of the cone.

28. The largest angle made by two elements of a solid aluminum cone is 60°. The elements are 10 inches long. A 2-inch hole is drilled through

the center of the cone so that the axis of the hole falls along the axis of the cone. How much of the metal is removed?

29. In a regular square pyramid a lateral edge which is 12.4 feet long makes an angle of 60° with the base. Find the volume of the pyramid.

30. In the quadrangular pyramid $V\text{-}ABCD$ the diagonals AC and BD of the base intersect at E. AC is extended to F, making $CF = AE$. Prove pyramid $V\text{-}BDF$ = pyramid $V\text{-}ABCD$.



31. The sides of the base of a triangular pyramid are 15, 41, and 52 inches. The altitude of the pyramid is 18 inches. Find the volume.

32. The total area of a right circular cone is 36π square inches, and its lateral area is 24π square inches. Find the volume of the cone.

33. The ratio of the numerical measures of the volume of a regular octahedron to its total surface is $\sqrt{6}:6$. Find the edge of the octahedron.

34. A wall is illuminated by a spotlight 18 feet away. A man 12 feet from the wall casts a shadow 15 square feet on the wall. What will be the area of the shadow if he moves 4 feet closer to the wall?

SUPERIOR WORK

S 1. The lateral faces of a certain square pyramid are equilateral triangles. Find the number of degrees in the dihedral angle between a lateral face and the base.

S 2. The lateral area of a right circular cone is twice the area of the base. Find the angle which an element makes with the altitude.

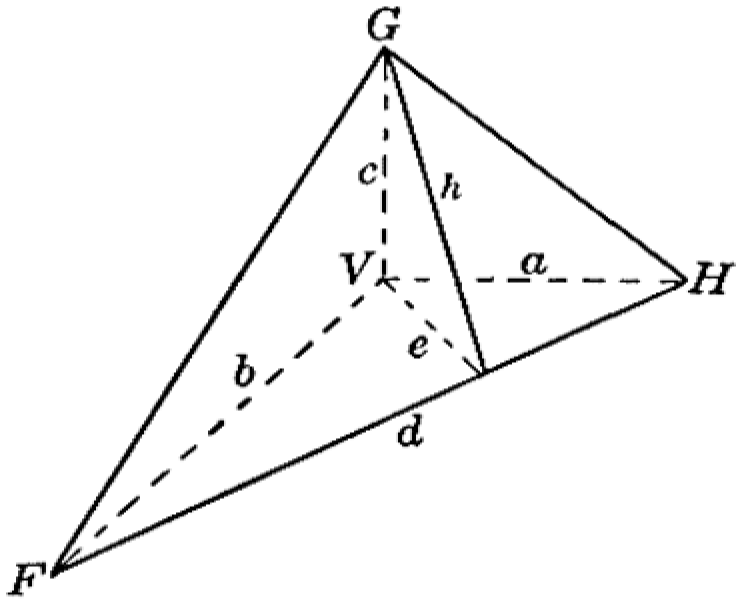
S 3. In a cone of revolution the altitude is h and the radius of the base is r. If $\frac{1}{h^2} + \frac{1}{r^2} = 1$, find the ratio of the numerical measures of the lateral area and the volume of the cone.

S 4. The lateral edge of a frustum of a regular quadrangular pyramid is 13 inches, the altitude of the frustum is 12 inches, and a side of the larger base is $8\sqrt{2}$ inches. Find the volume of the frustum.

S 5. Prove the following theorem which is analogous to the Pythagorean Theorem in plane geometry.

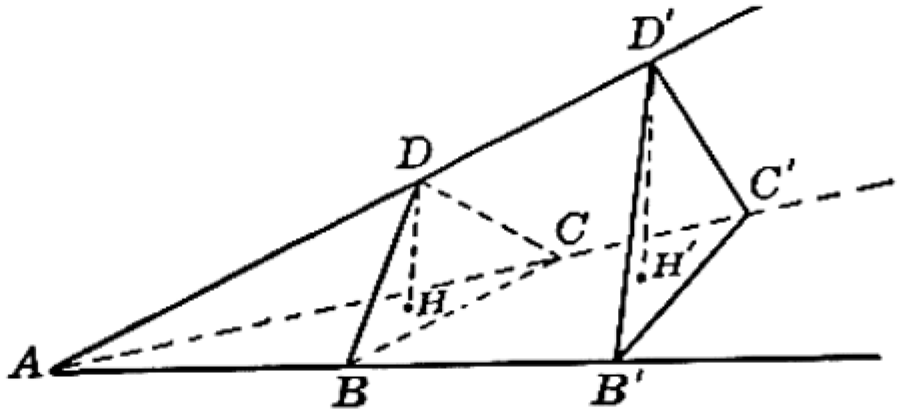
If the three lateral faces of a triangular pyramid are mutually perpendicular, the sum of the squares of the areas of the lateral faces equals the square of the area of the base.

Hint: Draw the figure with one lateral face VFH horizontal. Draw the altitude h of $\triangle FGH$, and altitude e of $\triangle VFH$. Prove: $d^2h^2 = a^2c^2 + b^2c^2 + d^2e^2$.



S 6. The following theorem is analogous to the proposition in plane geometry about two triangles that have an angle of one equal to an angle of the other (§ 478). Prove it.

THEOREM. If two tetrahedrons have the same trihedral angle, their volumes are proportional to the products of the three lateral edges of the trihedral angle.



Given: Tetrahedrons $ABCD$ and $AB'C'D'$ having trihedral $\angle A$ in common.

Prove:

$$\frac{ABCD}{AB'C'D'} = \frac{AB \times AC \times AD}{AB' \times AC' \times AD'}.$$

Help: Draw altitudes DH and $D'H'$.

$$\frac{ABCD}{AB'C'D'} = \frac{\triangle ABC \cdot DH}{\triangle AB'C' \cdot D'H'}.$$

$$\frac{\triangle ABC}{\triangle AB'C'} = \frac{AB \cdot AC}{AB' \cdot AC'}.$$

$$\frac{DH}{D'H'} = \frac{AD}{AD'}.$$

TEST

1. A spotlight is 45 feet from a screen. A plate whose area is 26 square inches is held parallel to the screen and 15 feet from it. How many square inches are in its shadow?

2. An open pail is 14 inches in diameter at the top and 10 inches at the bottom. The height is 12 inches. Find the area of the metal in square feet.

3. The altitude of a regular hexagonal pyramid is 4 inches and a side of its base is $2\sqrt{3}$ inches. Find the lateral area of a cone inscribed in the pyramid.

4. A semicircle of 12-inch radius is rolled into the lateral surface of a cone. Find the volume of the cone.

5. Two pyramids have the same base and their vertices on opposite sides of the base. A line segment connecting the vertices is divided by the plane of the base into two line segments. Prove that the volumes of the pyramids are proportional to these line segments.

6. The radius of the base of a cone is 5 inches. The axis of the cone, which is 16 inches, is inclined 30° to the base. Find the volume of the cone.

7. The volume of an octahedron is 10 cubic inches and two of its diagonals are 3 inches and 4 inches. Find the third diagonal if all three are mutually perpendicular.

8. **a.** The area of the base of a pyramid is 80 square inches. The altitude of the pyramid is 12 inches. A plane section of the pyramid parallel to the base has an area of 20 square inches. How far from the vertex is the section?
   b. Find the volume of the frustum of the pyramid in (a).

9. Which of the following statements are true and which are false?
   a. If the base of a pyramid is a regular polygon, the figure is a regular pyramid.
   b. In a regular pyramid the lateral faces are equilateral triangles.
   c. In a regular pyramid all the elements are equally inclined to the base.
   d. In a right circular cone all the elements are equally inclined to the base.
   e. A regular triangular pyramid is a regular tetrahedron.
   f. Some sections of an oblique cone are isosceles triangles.
   g. In a right circular cone the slant height, the altitude, and the radius of the base satisfy the Pythagorean relationship.
   h. If a triangle revolves about one of its sides as an axis, it generates a cone.
   i. If a plane is parallel to the base of a cone and bisects the altitude, it divides the cone into two equal solids.
   j. If two intersecting planes are tangent to the same cone, their intersection is parallel to some element of the cone.
   k. The areas of two sections parallel to the base of a pyramid are proportional to the distances of the sections from the vertex of the pyramid.
   l. The radii of two sections parallel to the base of a cone are proportional to the distances of the sections from the vertex of the cone.
   m. A triangular prism cannot be divided into three congruent pyramids.
   n. A cube can be divided into six congruent pyramids.
   o. If a pyramid is cut by a plane parallel to the base, the volumes of the two pyramids are proportional to the cubes of their altitudes.

10. Complete each of the following statements:

a. The two parts of a conical surface on either side of the vertex are called its —?—.

b. A pyramid has —?— diagonal(s).

c. Only a(n) —?— cone has a slant height.

d. The lateral surface of a pyramid consists of a number of —?—.

e. A tetrahedron is the same as a(n) —?— pyramid.

f. A regular tetrahedron in solid geometry is analogous to a(n) —?— in plane geometry.

g. A plane figure that can be rolled up to form the lateral surface of a cone is called a(n) —?—.

h. All the elements of a(n) —?— cone are equal.

i. If the altitude of a cone is not the same as its axis, the cone is called —?—.

j. If an isosceles triangle revolves about the bisector of its vertex angle, it generates a(n) —?—.

k. If a line revolves about a fixed point in a fixed line, always making the same angle with the fixed line, it generates a(n) —?—.

l. If a pyramid and a cone have equal bases and equal altitudes, they are —?—.

m. The lateral faces of a frustum of a pyramid are —?—.

n. In a regular octahedron, the diagonals —?— and —?—.

o. In a triangular truncated pyramid no faces are —?—.

5 Rotating a Semicircle

Properties and Measurement of the Sphere

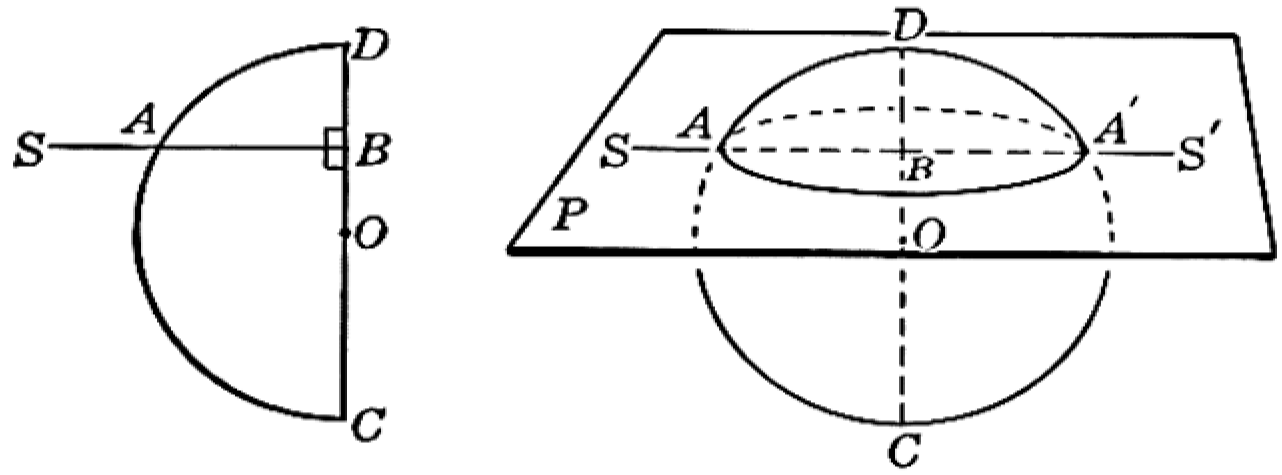
THE SPHERE AND ITS CIRCLES

244. The surfaces which we have studied so far can be generated by moving a straight line. We shall now study the surface that is generated by revolving a semicircle about its diameter. This surface is called a *sphere.*

245. Definitions. *A sphere is a surface generated by revolving a semicircle about its diameter. The center of the sphere is the center of the generating semicircle. The radius of the sphere is a line segment from the center to any point on the sphere.*

A chord of a sphere is a line segment connecting two points on the sphere.

A diameter of a sphere is a chord passing through the center of the the sphere.



As semicircle DAC in the figure revolves about its diameter DC,

(*a*) it generates the sphere O.

(*b*) any line SB perpendicular to DC generates a plane P perpendicular to DC.

(*c*) any point A on the semicircle generates a circle whose center is B and which is the intersection of P with sphere O.

246. Postulate. *One, and only one, sphere can be generated by a semicircle having a given center and a given radius.*

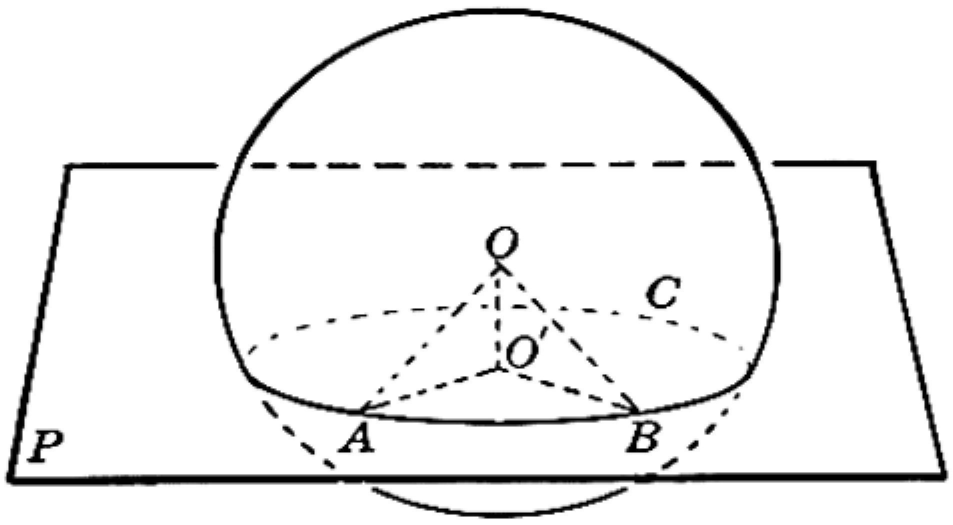
247. Corollaries

All radii or diameters of the same sphere are equal.

Spheres having equal radii are equal.

A sphere is the locus of a point in space that is a given distance from a given point.

248. THEOREM. If a plane intersects a sphere, the intersection is a circle.



Given: Section ABC is the intersection of plane P with sphere O.

Prove: Section ABC is a $\odot$.

Cons.: From O, the center of the sphere, draw $OO' \perp P$. Choose A and B, any two points on section ABC. Draw OA, OB, $O'A$, and $O'B$.

Proof: *Statements*	*Reasons*
1. $OA = OB$.	1. Radii of same sphere =.
2. $O'A = O'B$.	2. From a point in a perpendicular to a plane, = line segments meet the plane at = distances from the foot of the perpendicular. (§ 38)
3. Since all points of section ABC are equidistant from O', ABC is a circle.	3. Def. $\odot$.

249. Definitions. *A great circle of a sphere is a circle on the sphere whose plane passes through the center. All other circles on the sphere are called small circles.*

The axis of a circle of a sphere is the diameter of the sphere which is perpendicular to the plane of the circle. The ends of the axis are called the poles of the circle.

In the figure of § 245, DC is the axis of circle B. D and C are the poles of circle B.

Corollaries

250. The axis of a circle of a sphere passes through the center of the circle.

251. A diameter of a sphere passing through the center of a small circle is the axis of the circle.

252. All great circles of the same sphere are equal.

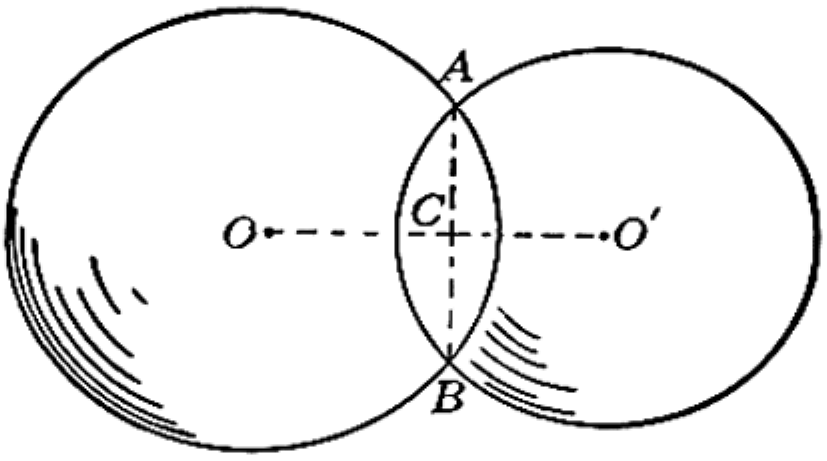
253. Three points on a sphere determine a circle.

254. Two points on a sphere which are not the ends of a diameter determine a great circle of the sphere.

255. A great circle bisects a sphere.

256. Definition. *A hemisphere is one-half of a sphere.*

257. THEOREM. The intersection of two spheres is a circle.



Given: Spheres O and O' intersecting.

Prove: The intersection of spheres O and O' is a circle.

Proof: *Statements*	*Reasons*
1. A plane through O and O' intersects the spheres in circles O and O'.	1. The intersection of a plane and a sphere is a circle. (§ 248)
2. The line of centers OO' bisects AB, the common chord of circles O and O', at right angles.	2. If two circles intersect, the line of centers bisects the common chord at rt. $\angle$s. (§ 449)
3. The given spheres may be generated by rotating circles O and O' about OO'.	3. One, and only one, sphere may be generated by a semicircle having a given center and a given line segment as a radius. (§ 246)
4. AB will generate a plane perpendicular to OO'.	4. All the lines $\perp$ to a given line at a given point lie in a plane $\perp$ to the line at that point. (§ 35)
5. The intersection of spheres O and O' is a circle.	5. Same as 1.

258. Corollary. If two spheres intersect, their line of centers passes through the center of the circle of intersection and is perpendicular to the plane of that circle.

EXERCISES

1. Prove that if two circles of a sphere have a common pole, they are in parallel planes.

2. Prove that if two circles of a sphere are in parallel planes, they have the same axis and poles.

3. Prove: A small circle of a sphere is determined by any point on it and its pole.

4. Prove that the intersections of two great circles of a sphere are the ends of a diameter of the sphere.

5. Prove that any two great circles of a sphere bisect each other.

6. Prove that two great circles of a sphere cannot have a common pole.

7. Prove: If the planes of two great circles of a sphere are perpendicular to each other, each circle passes through the poles of the other.

8. Prove: If one great circle of a sphere passes through the pole of another great circle, then the second circle passes through the pole of the first.

9. Prove: If the vertices of a triangle are on a sphere and two of them are diametrically opposite, the triangle is a right triangle.

10. Prove that if the diameter of a sphere is perpendicular to a chord, it bisects the chord.

11. Prove: The plane that bisects a chord of a sphere perpendicularly passes through the center of the sphere.

12. Prove: In the same sphere or in equal spheres, if two chords are equal, they are equally distant from the center.

13. Prove the converse of Exercise 12.

14. Prove: If two circles of a sphere are equal, their planes are equally distant from the center of the sphere.

15. Prove the converse of Exercise 14.

16. Find the locus of
  a. the center of a sphere that passes through two given points.
  b. a point that is a given distance from a given point and another given distance from a given plane.
  c. a point on a sphere equidistant from two intersecting planes.
  d. a point on a sphere equidistant from the ends of a given diameter.
  e. a point on a sphere at a given distance from a given point.

17. Find the locus of
  a. a point at a given distance from a given sphere.
  b. a point at a given distance from one fixed point and another given distance from a second fixed point.
  c. a point that is a given distance from a fixed line and another given distance from a point on the line.
  d. the center of a small circle of a sphere if the radius of the circle is constant.

e. the vertex of the right angle of a right triangle if the hypotenuse is fixed.

18. The radius of a sphere is 10 inches. Find the area of a circle of the sphere that is 8 inches from the center of the sphere.

19. The area of a circle of a sphere is 25π square inches. How far is this circle from the center of the sphere if the diameter of the sphere is 26 inches?

20. A circle of a sphere is 24 inches from the center and it has a circumference of 44 inches. Find the area of another circle of the sphere that is 20 inches from the center. (Use $\pi = \frac{22}{7}$.)

21. The base of a cone is a circle of a sphere and the vertex is at the center of the sphere. If the circumference of the base of the cone is 10π inches and the radius of the sphere is 13 inches, find the volume of the cone.

22. A right circular cone has its vertex at the center of a sphere of radius 10 inches. The altitude of the cone is 8 inches, and the largest angle between two elements is 90°. Find the area of the part of the base of the cone which is outside the sphere.

23. **a.** How do you pass a great circle through two given points on a sphere?
- **b.** How many great circles can pass through two given points on a sphere if these points are diametrically opposite? if they are not?
- **c.** How many small circles can pass through two points on a sphere if the two points are the ends of a diameter? if they are not?
- **d.** Through a given point on a sphere how many great circles have their planes parallel to the plane of a given great circle? to the plane of a given small circle?

24. The axis of the earth is the diameter about which the earth rotates. The ends of the axis are called the *poles*.
- **a.** How many great circles can pass through the north and south poles of the earth? These circles are called *meridians*.
- **b.** How many meridians can pass through a point not a pole on the earth's surface?
- **c.** Is the equator a great or a small circle? All the circles on the earth whose planes are parallel to the plane of the equator are called circles of *latitude*. Are the circles of latitude great or small circles?
- **d.** How many circles of latitude can pass through a given point on the earth's surface? Why?
- **e.** If you travel east, what kind of circle are you following?
- **f.** What is the name of the circle that you follow when you go directly south?

25. Prove that if a plane cuts two concentric spheres, the intersections are concentric circles.

26. Prove that if two chords intersect within a sphere, the product of the segments of one chord equals the product of the segments of the other chord.

27. If one side of an equilateral triangle is a diameter of a sphere, prove that the sphere bisects the other two sides.

28. The distances from the poles of a circle of a sphere to the circle are 7 inches and 5 inches respectively. Find the area of the circle.

29. a. It is required to draw a great circle on a blackboard globe 12 inches in diameter by placing one point of the compass at the pole. What should be the distance between the compass points?

b. If this circle were to be drawn by using a string tacked to the pole, how long would the string have to be?

30. Prove: The midpoint of a chord of a sphere is equidistant from the ends of any other chord parallel to the bisected chord.

31. The radii of two spheres are 9 inches and 12 inches. Their line of centers is 15 inches. Find the circumference of the circle of intersection of the two spheres.

32. Two spheres of radii 9 inches and 10 inches have their centers 17 inches apart. Find the circumference of the circle in which the spheres intersect. (Use § 476.)

33. Two spheres of radii 5 inches and 10 inches have their centers $5\sqrt{5}$ inches apart. Find the area of the circle which is the intersection of the spheres.

34. Prove that the product of the collinear line segments from a point to a sphere is constant.

35. A plane contains a chord of a sphere. As the plane rotates about this chord it cuts varying circles on the sphere. Find the locus of the centers of these circles.

36. a. Prove: If the planes of two circles of a sphere are unequally distant from the center of the sphere, the circles are unequal and the larger circle is nearer the center.

Hint: Use the method of § 39.

b. Prove the converse of (a) by the indirect method.

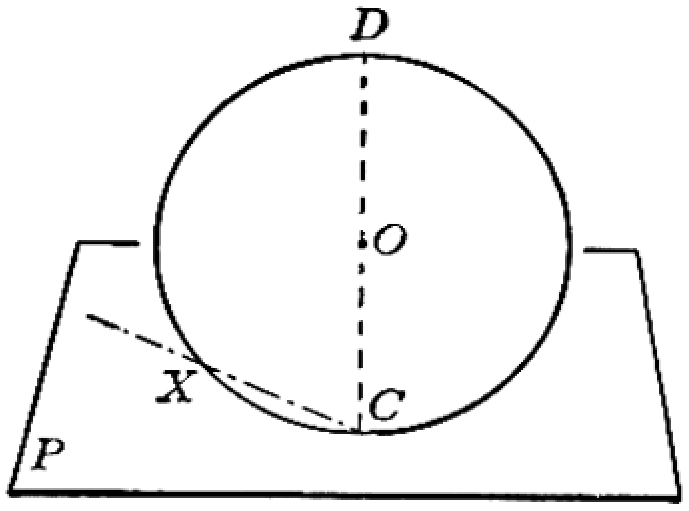
INSCRIBED AND CIRCUMSCRIBED SPHERES

259. Definition. *A line or a plane is tangent to a sphere if it touches the sphere in only one point called the point of contact or the point of tangency.*

260. Corollary. If a sphere is tangent to a plane, it is tangent to any line in the plane passing through the point of contact.

261. Definition. *Two spheres are tangent to each other if they are tangent to the same plane at the same point.*

262. THEOREM. A plane perpendicular to a diameter of a sphere at one end is tangent to the sphere.



GIVEN: Plane $P \perp DC$, diam. of sphere O, at C.

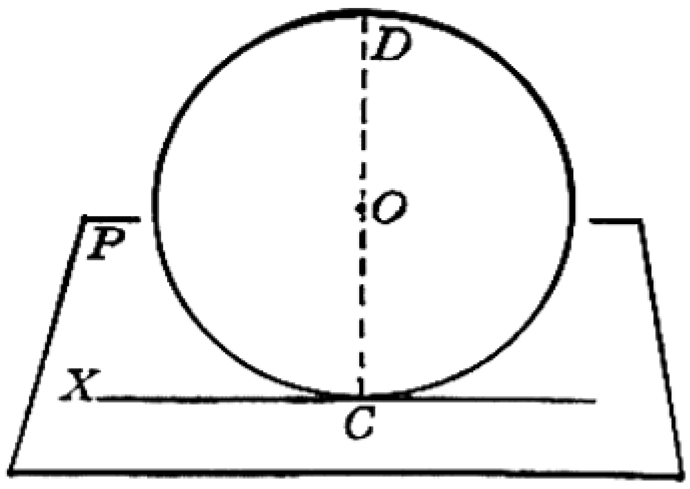
PROVE: P tan. to sphere O.

INDIRECT PROOF:

Statements	*Reasons*
1. Either P is tan. to sphere O or not.	
2. *Assume* that P is not tan. to sphere O; then it will cut sphere O in some pt. X.	2. Def. plane tan. to sphere.
3. Pass a plane through DC and X, cutting sphere O in $\odot DCX$ and plane P in CX.	3. The intersection of a plane and a sphere is a circle.
4. $CX \perp DC$.	4. Def. line $\perp$ to plane.
5. CX tan. to $\odot DCX$.	5. A line $\perp$ to diam. at one end is tan. to $\odot$. (§ 515)
6. This contradicts the def. of a line tan. to a $\odot$.	
7. Hence, P is tan. to sphere O.	

263. Corollary. If a sphere is tangent to two lines at their point of intersection, it is tangent to the plane of these lines.

264. CONVERSE THEOREM. If a plane is tangent to a sphere at the end of a diameter, it is perpendicular to the diameter.



GIVEN: Plane P tan. to sphere O at C.
DC diam. of sphere O.

PROVE: $DC \perp P$.

CONS.: Draw CX, *any* line in P through C. Pass a plane through DC and CX cutting sphere O in circle O.

PROOF: *Statements*	*Reasons*
1. CX is tan. to sphere O.	1. If a sphere is tan. to a plane, it is tan. to any line in the plane through the pt. of contact. (§ 260)
2. CX is tan. to circle O.	2. Def. line tan. to a circle.
3. $DC \perp CX$.	3. If a line is tan. to a $\odot$ at the end of a diam., it is $\perp$ to diam.
4. $DC \perp P$.	4. Def. line $\perp$ to plane.

265. Corollary. If two spheres are tangent to each other, the line of centers passes through the point of contact.

Draw the radii to the point of contact and prove that they are in a straight line. You can prove this indirectly by assuming that the radii are not in a straight line.

Definitions

266. *A sphere is inscribed in a polyhedron if it is tangent to all the faces of the polyhedron. The polyhedron is then circumscribed about the sphere.*

267. *A sphere is circumscribed about a polyhedron if it contains all the vertices of the polyhedron. The polyhedron is then inscribed in the sphere.*

268. *A sphere is inscribed in a cylinder, a cone, or a frustum of a cone if it is tangent to all the elements and the bases. These figures are then circumscribed about the sphere.*

269. *A sphere is circumscribed about a cylinder, a cone, or a frustum of a cone if all the elements are chords of the sphere. These figures are then inscribed in the sphere.*

270. Corollary. A sphere may be inscribed in any tetrahedron.

In § 107 it was proved that there is a point that is equidistant from the four faces of a tetrahedron. Describe the construction.

271. Corollary. A sphere may be circumscribed about any tetrahedron.

In § 102 it was proved that there is a point equidistant from the four vertices of a tetrahedron. Describe the construction.

EXERCISES

1. Through a given point on a sphere construct a plane tangent to the sphere.

2. Prove: If two planes are tangent to a sphere at the ends of a diameter, they are parallel.

3. Prove: If a line is tangent to a sphere, it is perpendicular to the diameter of the sphere at the point of contact.

4. Prove: If a line is perpendicular to a diameter of a sphere at one end, it is tangent to the sphere.

5. Prove: If a line is tangent to a circle of a sphere, it is tangent to the sphere. (Page 28, Exercise 9.)

6. Prove: If a line and a plane are tangent to a sphere at the same point, the plane contains the line.

7. Prove: If a line is tangent to a sphere, it is tangent to all the circles of the sphere that are in the same plane with the line.

8. Prove: If a plane is tangent to a sphere:
   a. a line perpendicular to the plane through the point of contact passes through the center of the sphere.
   b. a line perpendicular to the plane from the center of the sphere passes through the point of contact.

9. Prove: If a sphere is tangent to two parallel planes, the points of contact and the center of the sphere are collinear.

10. Prove: If a plane is tangent to the inner of two concentric spheres, the point of contact is the center of the circle in which the plane intersects the outer sphere.

11. Prove: If the diagonals of a quadrilateral inscribed in a sphere bisect each other, the quadrilateral is a rectangle.

12. Prove that a trapezoid inscribed in a sphere is isosceles.

13. Prove: If the sides of an isosceles triangle are tangent to a sphere, the point of contact on the base is the midpoint of the base.

14. A right circular cone is inscribed in a sphere. Prove that the vertex of the cone is a pole of the base.

15. A sphere of 10-inch radius is circumscribed about a cylinder whose axis section is equilateral. Find the total area of the cylinder.

16. A right circular cylinder whose altitude is 8 inches is inscribed in a sphere of radius 7 inches. Find the volume of the cylinder.

17. A right circular cone whose altitude is 24 inches is inscribed in a sphere of radius 15 inches. Find the volume of the cone.

18. An equilateral cone whose slant height is 12 inches is inscribed in a sphere. Find the diameter of the sphere. (An equilateral cone is one whose axis section is equilateral, § 215.)

19. Which of the following statements are true and which are false?

- **a.** At a given point on a sphere any number of lines can be tangent to the sphere.
- **b.** Two great circles of a sphere cannot have a common tangent line.
- **c.** If two lines are tangent to a sphere at the ends of a diameter, they are parallel.
- **d.** An equilateral quadrilateral inscribed in a sphere is a square.
- **e.** If a line is tangent to a circle of a sphere, it is perpendicular to the diameter of the sphere at the point of contact.
- **f.** If two spheres are tangent to the same line at the same point, they are tangent to each other.

20. Under what conditions do four points determine a sphere?

21. **a.** Prove: Lines from an external point that are tangent to a sphere are equal and make equal angles with the line joining the point to the center of the sphere.

- **b.** What is the locus of the points of contact in (a)?

22. Prove that the volume of a tetrahedron equals one-third the product of the area of the tetrahedron and the radius of the inscribed sphere.

23. Prove: If two intersecting planes are tangent to a sphere, the points of contact and the center of the sphere determine a plane perpendicular to the intersection of the tangent planes.

24. Prove that a sphere can be inscribed in a cube.

25. Prove that a sphere can be circumscribed about a rectangular solid.

26. **a.** Under what conditions can a sphere be inscribed in a cylinder? in a cone?

- **b.** Under what conditions can a sphere be circumscribed about a cylinder? about a cone?

27. **a.** Can a sphere be inscribed in a pyramid? in a triangular pyramid? in a regular pyramid?

- **b.** Can a sphere be circumscribed about a pyramid? about a triangular pyramid? about a regular pyramid?

28. Find the locus of the center of a sphere of given radius tangent to a given plane; tangent to a given line.

29. Find the locus of the center of a sphere tangent to a given plane at a given point; tangent to a given line at a given point.

30. A sphere passes through a fixed point and is tangent to two fixed parallel planes. Find the locus of the center of the sphere.

31. **a.** Find the locus of the center of a sphere that passes through three given points.

- **b.** Given its radius, construct a sphere that will pass through three given points.

32. If two spheres are tangent to each other, prove that any plane through their centers passes through the point of contact.

33. Prove: If two spheres are tangent internally, chords of the larger sphere from the point of contact are divided in a constant ratio by the smaller sphere.

34. a. Prove: If a diameter of one sphere is the radius of another sphere, any chord of the larger sphere from the point of contact is bisected by the smaller sphere.
    b. If a chord rotates about a stationary point on a sphere, find the locus of its midpoint.

35. Through the point of tangency of two spheres two lines are drawn forming four chords of the spheres. Prove that these chords are proportional.

36. Prove: If a sphere is inscribed in a cylinder or a cone, a plane that is tangent to the cylinder or the cone is also tangent to the sphere.

37. Prove: The radius of a sphere inscribed in a regular triangular pyramid is one-fourth the altitude of the pyramid, and the radius of the circumscribed sphere is three-fourths the altitude. (See page 38, Exercise 21.)

38. Through a given external point construct a plane tangent to a given sphere.

39. Prove that lines drawn tangent to two spheres from a point in the plane of their circle of intersection are equal.

40. A sphere is inscribed in a frustum of a right circular cone. Find the radius of the sphere if the radii of the bases of the frustum are 2 inches and 8 inches respectively.

SUPERIOR WORK

S 1. What is the largest number of spheres that can be determined by six points?

S 2. Two steel balls 10 centimeters and 4 centimeters in diameter are to rest on the bottom of a cylindrical can. What is the smallest diameter that the can may have?

S 3. A regular pyramid having a base 6 inches square is inscribed in a sphere of 10-inch radius. Find the volume of the pyramid.

S 4. A frustum of a right circular cone is inscribed in a sphere of 25-inch radius. The radii of the bases of the frustum are 20 inches and 24 inches respectively. Find the volume of the frustum. (Two answers are possible.)

S 5. A right circular cone whose slant height is $8\sqrt{5}$ inches is inscribed in a sphere of radius 10 inches. Find the volume of the cone.

S 6. Given its radius, construct a sphere passing through two given points and tangent to a given plane.

S 7. A sphere of 5-inch radius is inscribed in a right circular cone whose altitude is 18 inches. Find the lateral area of the cone.

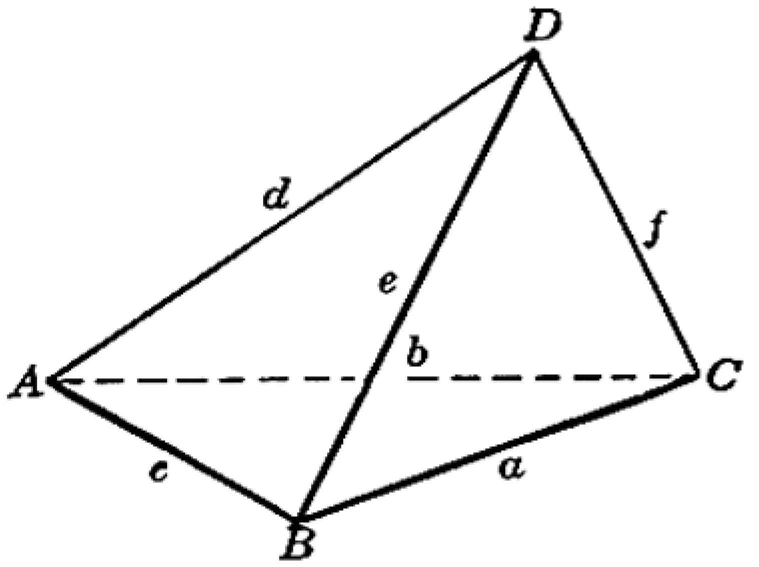
S 8. Construct a plane tangent to a given sphere and containing a given line. *Hint:* Use the skew quadrilateral theorem of § 25.

S 9. A sphere of radius 10 inches is tangent to two parallel planes. A right circular cone has its base in one of the planes and its vertex in the other plane. The base of the cone equals a great circle of the sphere. How far from the vertex of the cone is a plane that cuts equal circles from the sphere and the cone?

S 10. a. Given any triangle, it is possible to construct three circles tangent to one another having the vertices of the triangle as centers. Given a tetrahedron, it is not always possible to construct four spheres tangent to one another having the vertices of the tetrahedron as their centers. Prove that if this can be done, the sum of two opposite edges of the tetrahedron equals the sum of two other opposite edges.

Help: Let radius of sphere $C = x$.
Then radius of sphere $B = a - x$
and radius of sphere $A = b - x$
and radius of sphere $D = f - x$
$= e - (a - x)$
$= d - (b - x)$.

b. In the figure $a = 5$, $b = 6$, $c = 9$, $d = 8$, $e = 7$, and $f = 4$. Is it possible to construct spheres tangent to one another with A, B, C, and D as centers? If so, what are the radii of the spheres?

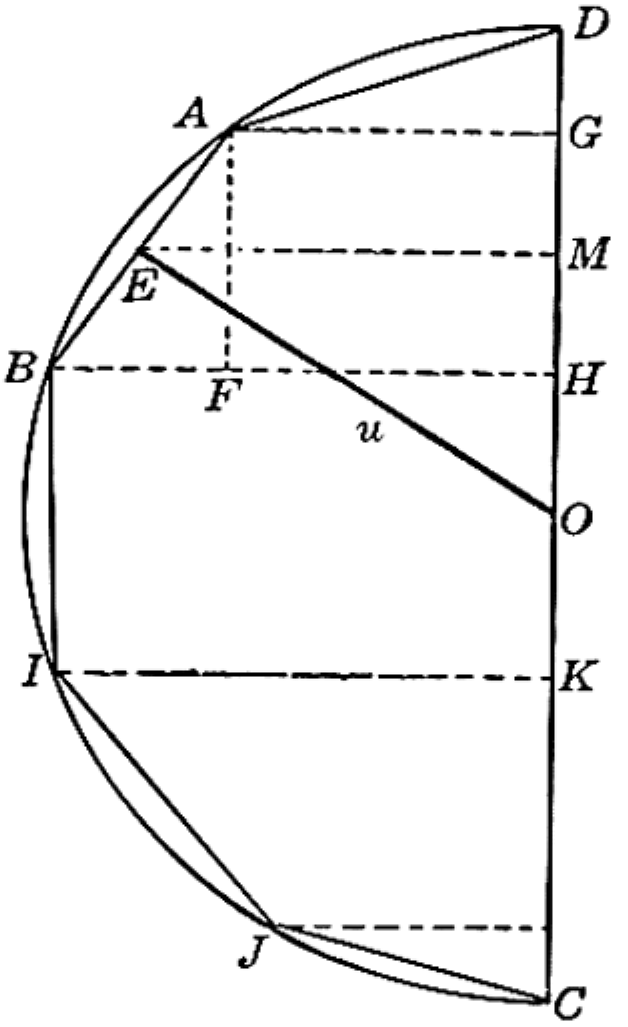


c. Four spheres are tangent to one another. Four of the lines of centers are 4, 6, 9, and 7. If the lines 4 and 7 do not intersect, find the lengths of the other two lines of centers.

AREA OF A SPHERE

272. As we have seen, a sphere is not a ruled surface. There is no way, therefore, to "roll out" a sphere into a flat surface. Nevertheless, the area of a sphere is *exactly* equal to the area of four of its great circles. To prove this, we shall find the area generated by revolving half a regular polygon of an even number of sides about its diameter. Then, as in plane geometry, we shall reason from the revolving semi-polygon to the revolving semicircle.

273. THEOREM. The area of the surface generated by half a regular polygon of an even number of sides inscribed in a semicircle equals the product of the diameter of the semicircle and the circumference of a circle whose radius is the apothem of the regular polygon.



GIVEN: $DAB \cdots C$ half a regular polygon inscribed in semicircle DBC.
u the apothem.
S = area generated by revolving $DAB \cdots C$ about diameter DC.

PROVE: $S = DC \cdot 2\pi u$.

CONS.: Draw GH, the projection on DC of AB.
From E, the midpoint of AB, draw $EM \perp DC$.
Let L = lateral area of the frustum of the cone generated by AB.

PLAN OF PROOF: Find L in terms of AB and EM. Then, from the similar triangles, replace these by u and GH.

PROOF: *Statements*	*Reasons*
1. $L=\frac{1}{2}\cdot AB(2\pi\cdot AG+2\pi\cdot BH)$. $L=2\pi\cdot AB\cdot\frac{1}{2}(AG+BH)$.	1. Lateral area of frustum of rt. circular cone = $\frac{1}{2}$ prod. slant height and sum of circumferences of bases. (§ 228)
2. $\frac{1}{2}(AG + BH) = EM$.	2. Line segment bet. midpts. of legs of trap. = $\frac{1}{2}$ sum of bases. (§ 499)
3. $L = 2\pi\cdot AB\cdot EM$.	3. Subs. Ax.
4. $\triangle ABF \sim \triangle EOM$.	4. If sides of one $\triangle$ are respectively $\perp$ to sides of another $\triangle$, $\triangle$s are $\sim$. (§ 479)
5. $\frac{AB}{EO} = \frac{AF}{EM}$. $AB\cdot EM = EO\cdot AF$. $AB\cdot EM = u\cdot GH$.	5. Def. $\sim$ polygons.

6. $L = 2\pi u \cdot GH$. Similarly, area generated by $DA = 2\pi u \cdot DG$ area generated by $BI = 2\pi u \cdot HK$, etc.	6. Subs. Ax. (in Eq. 3).
7. $S = 2\pi u(DG + GH + HK + \cdots)$ or $S = DC \cdot 2\pi u$	7. Add. Ax.

Note that chords *AD* and *JC* generate lateral surfaces of a cone, and one chord such as *BI* may generate the lateral surface of a cylinder. May these be regarded as special cases of a frustum of a cone? Why?

In plane geometry we assumed that any proposition about a regular polygon that does not depend on the number of sides is equally valid for the circumscribed circle (§ 439). We apply this to our semi-polygon and, recalling that the apothem u approaches the radius as a limit, we have the area generated by the circumscribed semicircle. This, of course, is the area of the sphere.

274. Corollary. The area of a sphere equals the product of the diameter of the sphere and the circumference of a great circle.

Thus, if S is the area of the sphere, d its diameter, and r its radius,

$$S = d \cdot \pi d = \pi d^2$$
$$S = \pi(2r)^2 = 4\pi r^2$$

275. Corollary. The area of a sphere equals four times the area of a great circle of the sphere.

276. Corollary. The areas of two spheres are to each other as the squares of their diameters or radii.

Zones

If a semicircle revolves about its diameter, it generates a sphere. A part of this semicircle generates a surface called a *zone*.

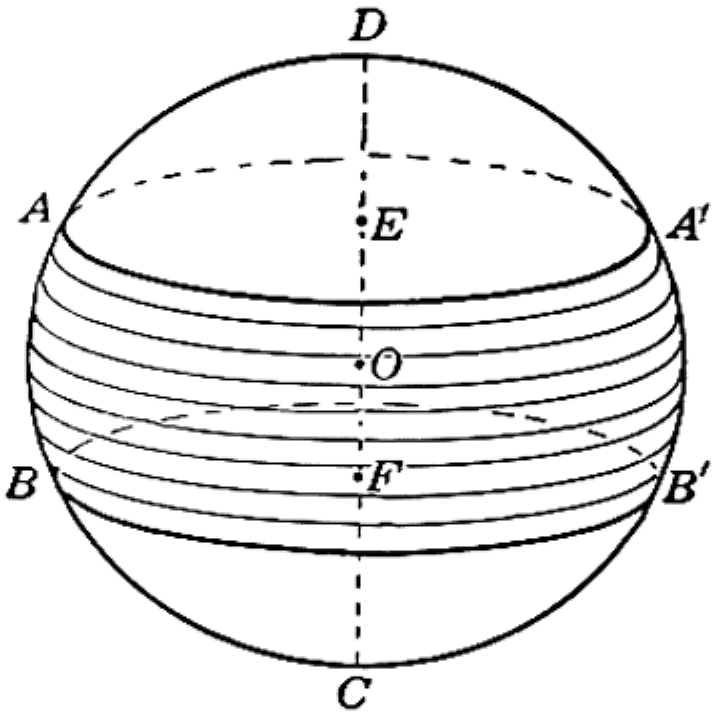
277. Definition. *A zone is a portion of a sphere generated by revolving a minor arc about a diameter.*

278. Corollary. A zone is a part of a spherical surface included between two parallel planes.

279. Definitions. *The circles that bound a zone are called the bases of the zone. The distance between the planes that include the zone is the altitude of the zone. If one of these planes is tangent to the sphere, the zone is called a zone of one base or a dome.*

In the figure arc AB generates a zone of two bases. The zone is the surface of the sphere between the circles whose centers are E and F. The altitude of this zone is EF.

Arc DA generates a zone of one base, circle E. The altitude of this zone is DE. The whole sphere is divided into three zones.



280. By referring to the figure for the theorem in § 273, it can be seen that arc AB generates a zone. The arc AB is the limit of the chords between A and B as their number is repeatedly doubled. Reasoning as we did about the semicircle, we shall arrive at the conclusion that the area generated by arc AB equals $GH \cdot 2\pi r$.

Letting $h = GH =$ altitude of the zone, the area of the zone $Z = 2\pi rh$.

281. Corollary. The area of a zone equals the product of its altitude and the circumference of a great circle.

282. Corollary. On the same sphere or on equal spheres the areas of two zones are to each other as their altitudes.

EXERCISES

1. Find the area of a sphere whose diameter is 10 inches.
2. Find the area of a hemisphere whose radius is 20 inches.
3. At $3.50 a gallon for paint what is the cost of painting a hemispherical dome 30 feet in diameter if a gallon covers 400 square feet?
4. Find the radius of a sphere whose area is 154 square inches.
5. Taking the mean diameter of the earth as 7920 miles, find the area of the earth's surface.
6. The area of the United States is 3,000,000 square miles. What per cent of the globe does our country cover?
7. The circumference of a great circle of a sphere is 11 inches. Find the area of the sphere.

8. The circumference of a great circle of a sphere is C. Find the area of the sphere in terms of C.

9. Find the area of a sphere circumscribed about a cube whose edge is 15 inches.

10. Prove that the area of a sphere equals the lateral area of a circumscribed cylinder.

11. The radii of two spheres are 3 inches and 4 inches. Find the ratio of their areas.

12. The areas of two spheres are in the ratio of $4:9$. If the radius of the smaller sphere is 6 inches, find the radius of the larger sphere.

13. The diameter of the moon is 2160 miles, whereas the mean diameter of the earth is 7920 miles. How many times larger is the surface of the earth than the surface of the moon?

14. If the diameter of a sphere is increased by 10 per cent, by what per cent is the area increased?

15. If the area of a sphere is to be decreased by 36 per cent, by what per cent should the radius be decreased?

16. Find the diameter of a sphere whose area equals the sum of the areas of two spheres 24 inches and 7 inches in diameter respectively. What relationship do your three diameters satisfy?

17. The radius of a sphere is 10 inches. Find the area of a zone on it whose altitude is 12 inches.

18. The area of a sphere is 36π square inches. Find the area of a zone of 1-inch altitude.

19. The surface of a sphere has an area of 400 square inches. Find the area of a zone whose altitude is 10 inches.

20. A hollow rubber ball fits snugly into a tin can. Two planes parallel to the bases of the can cut the figure. Prove that the area of the rubber between the parallel planes equals the area of the tin between these planes.

21. A semicircle revolves about its 12-inch diameter. Find the area generated by an arc whose projection on the diameter is 3 inches.

22. A 60° arc is revolved about its 10-inch diameter. The end of the arc and the end of the diameter are the same point. Find the area generated by the arc.

23. The arc that generates a dome (a zone of one base) is 30°. Find the area of the dome if its altitude is 6 feet.

24. On a sphere of radius 9 inches the area of a zone of one base is 36π square inches. Find the altitude of the zone.

25. A material ball 13 inches in diameter is cut into two parts by a plane 5 inches from the center. Find the total area of the larger part.

26. The radius of a sphere is 15 inches. A plane divides the diameter to which it is perpendicular in the ratio of 3:2. Find the areas of the two zones.

27. One base of a zone is a great circle. The arc generating the zone is 30°. Find the area of the zone if the radius of the sphere is 10 inches.

28. The diameter of one sphere is twice that of another sphere. If two zones, one on each sphere, have equal altitudes, what is the ratio of their areas?

29. Find the area of a sphere circumscribed about an equilateral cone whose slant height is 12 inches.
(In an equilateral cone the axis section is equilateral.)

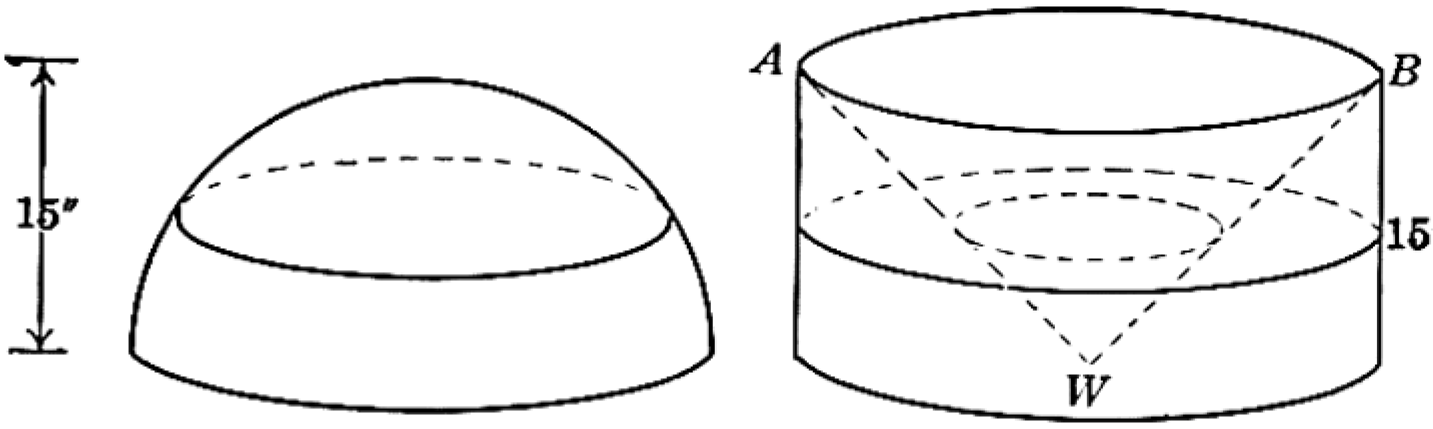
30. A sphere is inscribed in an equilateral cone. Find the ratio of the lateral area of the cone to the area of the sphere.

31. Find the area of a sphere inscribed in a regular hexagonal prism whose lateral area is 48 square inches.

32. A sphere is divided into four equal zones. Find the number of degrees in the generating arc of each zone.

33. Find the area of a zone of one base if the radius of this base is 3 inches and if the plane of the base is 4 inches from the center of the sphere.

34. The figure on the left is a hemisphere of radius 15 inches. The figure on the right is a cylinder from which the cone *WAB* of the same base and altitude as the cylinder has been removed. The height of the "hollowed" cylinder is 15 inches and the radius of its base is 15 inches. Both figures are included between parallel planes.



a. A plane parallel to the bases and 9 inches from the lower bases cuts both figures. Find the area of the circle cut from the hemisphere and the area of the ring cut from the hollowed cylinder.
b. Solve (a) if the cutting plane is 12 inches from the lower base.
c. Solve (a) if the cutting plane is 7 inches from the lower base.

35. A point of light is located 8 inches from a sphere of radius 12 inches. How many square inches of the sphere does it illuminate?

SUPERIOR WORK

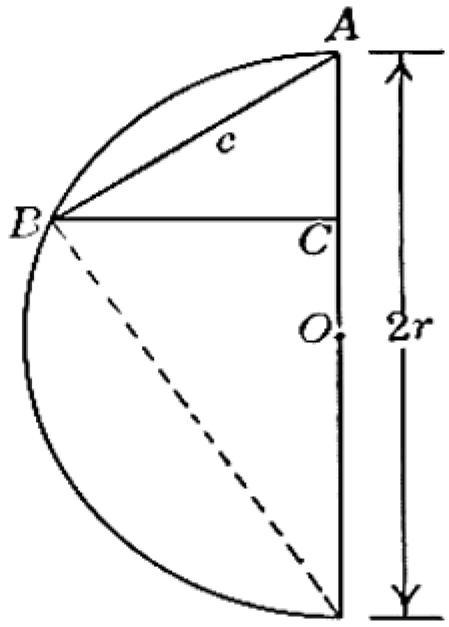
S 1. On a sphere of radius 15 inches the radii of the bases of a zone are 9 inches and 12 inches respectively. Find the area of the zone. (Two solutions are possible.)

S 2. The radii of the bases of a zone are 3 inches and 4 inches respectively. The altitude of the zone is 1 inch. Find the area of the zone.

S 3. Prove: The area of a zone of one base equals the area of a circle whose radius is the chord of the generating arc.

Help: In the figure prove that Z = area generated by $\widehat{AB} = \pi c^2$.

$$Z = AC \cdot 2\pi r.$$
$$c^2 = AC \cdot 2r. \quad (\S\ 489)$$



S 4. The base of a dome has a radius of 10 feet. The height of the dome is 4 feet. Find the area of the dome. (Use S 3.)

S 5. Prove: If the area of a dome is twice the area of its base, the dome is a hemisphere.

S 6. If the area of a dome is three times the area of its base, find the ratio of the altitude of the dome to the radius of the base.

S 7. *Problem:* Point A is a distance l from the sphere O whose radius is r. A light at A illuminates a fraction p of the sphere. Find the relation between l, p, and r.

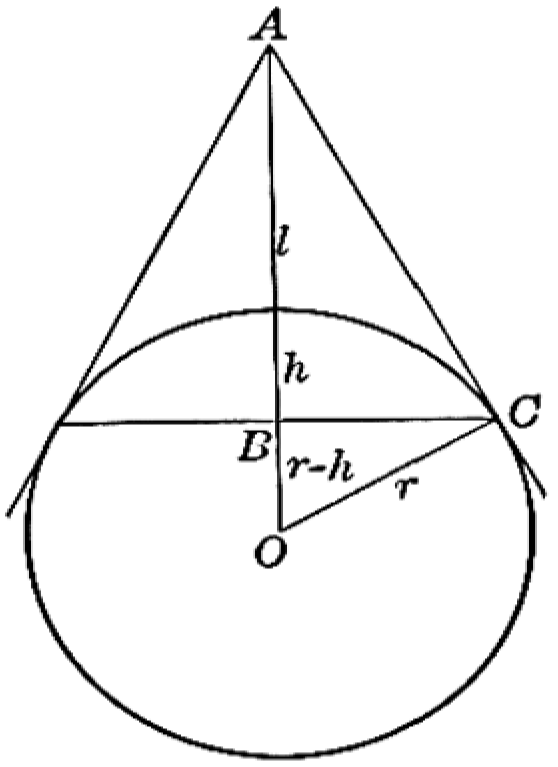
OUTLINE OF SOLUTION:

Let h = altitude of the illuminated zone.

$$OC^2 = OB \cdot OA. \quad (\S\ 489)$$
$$r^2 = (r - h)(r + l).$$
$$l = \frac{rh}{r - h}.$$

Since $p = \frac{h}{2r}$, (§ 282)

$$l = \frac{2pr}{1 - 2p}. \qquad (1)$$
$$p = \frac{l}{2(l + r)}. \qquad (2)$$



Write out the complete solution.

S 8. Use the results in S 7 to solve the following.

- **a.** How far from a sphere of radius 12 inches should a point of light be placed to illuminate one-third of the sphere?
- **b.** What part of a sphere 36 inches in diameter is illuminated by a point of light 6 inches from the sphere?

S 9. **a.** A point of light illuminates one-fourth of a sphere. Compare the distance from the point to the sphere with the radius of the sphere.

b. Formula (2) in S 7 may be written $p = \dfrac{1}{2\left(1 + \frac{r}{l}\right)}$.

What value does p approach as l increases indefinitely? What is the largest part of a sphere that can be seen from an external point?

S 10. **a.** Find the number of square miles that can be seen by a flier 2 miles above the earth.

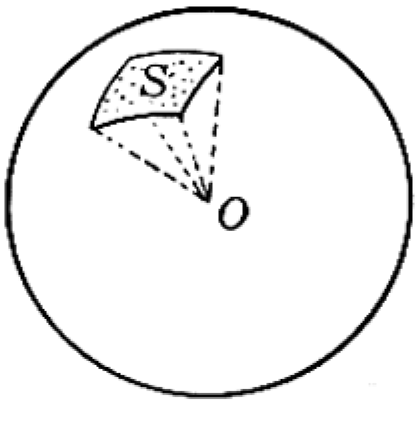
b. The United States Air Force has taken a picture from a V–2 rocket 100 miles above New Mexico. What part of the earth's surface could have been photographed from that point?

VOLUME OF A SPHERE

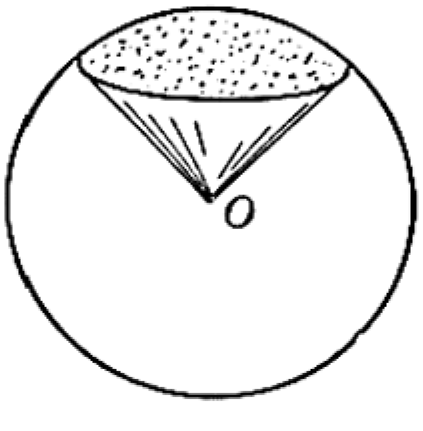
283. We shall now learn how to find the volume of a sphere and certain portions of it.

As a ray from the center of a sphere rotates about its origin, it generates a ruled surface (§ 116). The portion of a sphere bounded by this surface (or surfaces) and the sphere is a solid. We shall refer to it as a portion of the sphere originating at the center. The four important types of this solid are: the *spherical pyramid*, the *spherical cone*, the *spherical sector*, and the *spherical wedge*.

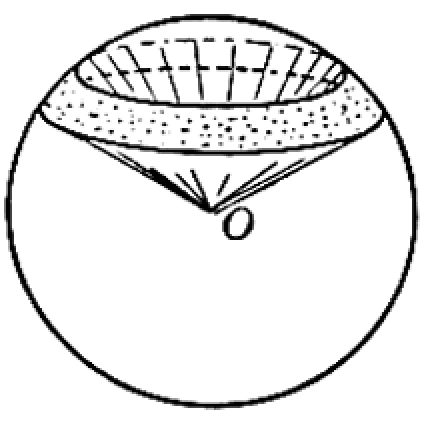
284. Definitions. *A spherical pyramid or a spherical cone is a solid bounded by a sphere and one nappe of a pyramidal or conical surface whose vertex is the center of the sphere. The base of the spherical pyramid or cone is the surface of the sphere included by the pyramidal or conical surface; the altitude is the radius of the sphere.*



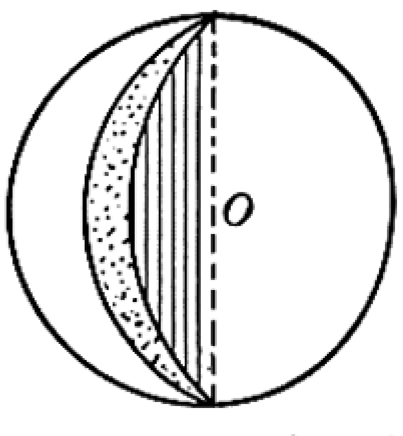
Spherical Pyramid



Spherical Cone



Spherical Sector



Spherical Wedge

285. Definitions. *A spherical sector is a solid generated by a sector of a semicircle as the semicircle is revolved about its diameter. The base of a spherical sector is the zone generated by the arc of the plane sector; the altitude is the radius of the sphere.*

286. Corollary. A spherical sector generated by revolving a plane sector about one of its radii is a spherical cone.

287. Definitions. *A spherical wedge is a solid bounded by the faces of a dihedral angle whose edge passes through the center of the sphere and by the portion of the sphere intercepted by those faces. The base of a spherical wedge is the surface of the sphere included between the faces of the dihedral angle; the altitude is the radius of the sphere.*

288. Definition. *The base of a spherical wedge is called a lune.*

Volume of a Spherical Pyramid

289. A spherical pyramid whose base is S, could be subdivided into many smaller pyramids whose bases s_1, s_2, s_3, . . . are plane surfaces. The more subdivisions, the more nearly would the sum of the pyramids with the plane bases approach the spherical pyramid, and the more nearly would the sum of the plane bases approach the spherical base. Since the sum of the small pyramids $= \frac{1}{3}s_1r + \frac{1}{3}s_2r + \frac{1}{3}s_3r + \cdots = \frac{1}{3}r(s_1 + s_2 + s_3 + \cdots)$, the volume of the spherical pyramid $= \frac{1}{3}Sr$.

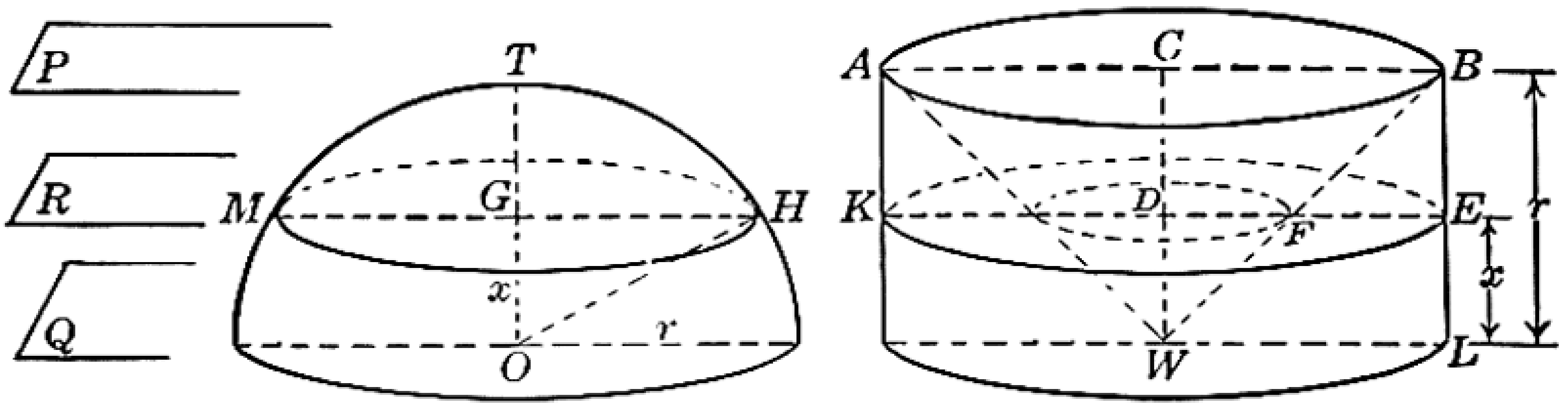
The same argument could be used for the other spherical solids originating at the center of the sphere. Furthermore, the whole solid sphere may be considered as the sum of a number of spherical pyramids. We therefore see that the volume of a sphere equals one-third the product of the area of the sphere and its radius. We are thus led to the following general theorem.

290. THEOREM. The volume of a sphere, a spherical pyramid, a spherical cone, a spherical sector, or a spherical wedge equals one-third the product of its spherical surface and the radius.

In the following sections parts of this theorem will be proved. Insofar as it concerns the whole sphere, the proof shows that a cylinder hollowed out by a cone has the same volume as a hemisphere because they are Cavalieri solids. (See page 160, Exercise 34.)

Note that $\frac{1}{3}Sr = \frac{1}{3} \cdot 4\pi r^2 \cdot r = \frac{4}{3}\pi r^3$.

291. THEOREM. The volume of a sphere of radius r equals $\frac{4}{3}\pi r^3$.



GIVEN: Hemisphere O with radius r.
V = volume of the whole sphere.

PROVE: $V = \frac{4}{3}\pi r^3$.

CONS.: Draw rt. circular cylinder AL having the radius of its base and its altitude each $= r$. From this cylinder remove the rt. circular cone AWB. Place the hemisphere and the hollowed cylinder between two $\parallel$ planes P and Q. Pass *any* third plane R at a distance x from Q, cutting the hemisphere in the $\odot MH$ and the hollowed cylinder in the ring FE.

PROOF: *Statements* Supply the reasons.

1. $GH^2 = r^2 - x^2$.
2. Area $\odot MH = \pi \cdot GH^2 = \pi(r^2 - x^2)$.
3. $DF = DW = x$.
   $DE = r$.
4. Area ring $FE = \pi \cdot DE^2 - \pi \cdot DF^2$,
   $= \pi r^2 - \pi x^2 = \pi(r^2 - x^2)$.
5. Area $\odot MH$ = area ring FE.
6. Volume hemisphere O = volume hollowed cylinder AL. (Cavalieri's Proposition § 176)
7. $\frac{V}{2}$ = Cylinder AL − Cone AWB.
   $= \pi r^3 - \frac{1}{3}\pi r^3 = \frac{2}{3}\pi r^3$.
8. $V = \frac{4}{3}\pi r^3$.

292. Corollary. The volume of a sphere equals one-third the product of the area of its surface and its radius.

293. Corollary. The volume of a sphere whose diameter is d equals $\frac{1}{6}\pi d^3$.

294. Corollary. The volumes of two spheres are proportional to the cubes of their radii or diameters.

The Spherical Segment

295. Definitions. *A spherical segment is a solid bounded by a sphere and two parallel planes intersecting or tangent to the sphere. The bases of a spherical segment are the intersections of the parallel planes with the solid bounded by the sphere; the altitude is the distance between the planes. If one of the planes is tangent to the sphere, the segment is a spherical segment of one base; otherwise it is a spherical segment of two bases.*

Volume of a Spherical Segment (Optional)

296. In the figure for the proof of the theorem § 291 the portion of the hemisphere above the plane R is a spherical segment of one base. From the proof of that theorem it is evident that this spherical segment and the part of the hollowed cylinder above plane R are Cavalieri solids and are therefore equal in volume. Call this volume K. Let GT, the altitude of the spherical segment, equal h.

1. $K = \text{cylinder } AE - \text{frustum } AF.$

$$= \pi r^2 h - \frac{h}{3}(\pi r^2 + \pi \cdot DF^2 + \sqrt{\pi r^2 \cdot \pi \cdot DF^2}), \quad (\S\ 240)$$

$$= \pi r^2 h - \frac{h}{3}(\pi r^2 + \pi \cdot DF^2 + \pi r \cdot DF),$$

$$= \frac{2}{3}\pi r^2 h - \frac{\pi h}{3} \cdot DF^2 - \frac{\pi h r}{3} \cdot DF.$$

2. $DF = DW = r - h.$

3. $K = \frac{2}{3}\pi r^2 h - \frac{\pi h}{3}(r - h)^2 - \frac{\pi h r}{3}(r - h),$

$$= \frac{3\pi h^2 r}{3} - \frac{\pi h^3}{3} = \frac{\pi}{3} h^2(3r - h).$$

Work out the algebraic solution indicated above. You will then have proved the following theorem.

297. THEOREM. If r is the radius of a sphere and h is the altitude of a spherical segment of one base, then the volume of the segment equals $\frac{\pi}{3} h^2(3r - h)$.

The volume of a spherical segment of two bases is usually found by subtracting two spherical segments of one base. A formula for the volume of a spherical segment of two bases appears on page 171, S 6.

Volume of a Spherical Cone (Optional)

In the figure of § 291,

1. Spherical cone $OMTH$ = spherical segment MTH + cone OMH
$$= \frac{\pi h^2}{3}(3r - h) + \frac{\pi \cdot GH^2}{3} \cdot OG. \quad (§ 297, § 234)$$
2. $OG = r - h.$
$GH^2 = OH^2 - OG^2 = r^2 - (r - h)^2 = 2rh - h^2.$
3. Spherical cone $OMTH = \frac{\pi h^2}{3}(3r - h) + \frac{\pi}{3}(2rh - h^2)(r - h),$
$$= \tfrac{2}{3}\pi r^2 h = \tfrac{1}{3}\cdot 2\pi rh \cdot r.$$
4. Let Z = area of zone $MTH = 2\pi rh.$ (§ 281)
5. Spherical cone $OMTH = \frac{1}{3}Zr.$

Work out the algebraic solution indicated above. You will then have proved the following theorem which we arrived at intuitively in § 289.

298. THEOREM. The volume of a spherical cone equals one-third the product of its base and altitude.

The volume of a spherical sector may be found by subtracting two spherical cones.

Volume of a Spherical Wedge

As a semicircle rotates about its diameter through a given dihedral angle it generates a lune and its interior generates a spherical wedge (§ 287, § 288). Since a rotation of 360° generates the entire sphere, a spherical wedge is the same part of the solid sphere as its dihedral angle is of 360°. The same comparison can be made between a lune and the entire surface of the sphere.

299. Postulate. *The volume of a spherical wedge is to the volume of its sphere, and the area of its lune is to the area of the sphere as the dihedral angle of the wedge is to 360°.*

EXERCISES

1. The radius of a sphere is $3\frac{1}{2}$ inches. Find its volume.

2. Find the volume of a hemisphere whose diameter is 42 inches.

3. Using 7920 miles as the mean diameter, find the volume of the earth.

4. a. The diameter of the planet Jupiter is about 10 times the diameter of the earth. How many times the volume of the earth is the volume of Jupiter?

b. The diameter of the sun is 10 times the diameter of Jupiter. Compare the volume of the sun with that of the earth.

5. A metal ball 1 foot in diameter weighs 300 pounds. Find the weight of a ball of the same metal 3 feet in diameter.

6. A gasoline storage tank in the form of a sphere is 40 feet in diameter. What is its capacity in gallons?

7. Find the volume of a sphere if one of its great circles is 11 inches long.

8. The area of a sphere is $17\frac{1}{9}$ square feet. Find its volume.

9. The area of a sphere is S. Find its volume in terms of π and S.

10. The inner surface of a spherical steel shell is 30 square inches, and the outer surface is 40 square inches. Find the volume of the steel.

11. The volume of a sphere is $14\frac{1}{7}$ cubic inches. Find its radius.

12. The radii of two spheres are in the ratio 5:6. Find the ratio of the volumes of the spheres.

13. The ratio of the volumes of two spheres is 27:64. Find the ratio of the radii.

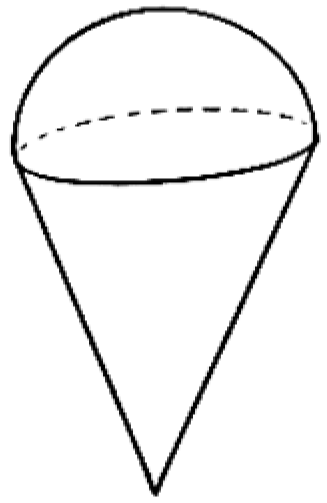
14. The areas of two spheres are to each other as 4 is to 9. How do their volumes compare?

15. Taking the diameter of the earth as 8000 miles and that of the moon as 2160 miles, find the ratio of the volumes of the two bodies.

16. There are 500 yards of yarn in a 5-inch ball. How many yards of yarn are in a 6-inch ball?

17. It is estimated that almost half the atmosphere is compressed below the elevation of 3 miles. Find the volume of this compressed air.

18. A cone 4 inches deep and 2 inches in diameter is packed with ice cream so that the top forms a hemisphere. Find the number of cubic inches of ice cream.



19. If 5-inch grapefruits sell for one dollar a dozen, what should be the cost of 4-inch grapefruits?

20. A hollow hemispherical dome has an outer diameter of 40 feet. Find the volume of the bricks and mortar if the dome is 10 inches thick.

21. Find the ratio of the volume of a sphere to the volume of the circumscribed cylinder.

22. Find the diameter of a hemispherical bowl that holds 1 gallon.

23. Find the radius of a sphere whose volume and area have the same numerical value.

24. Prove that the ratio of the total area of a right circular cylinder to that of the inscribed sphere equals the ratio of their volumes.

25. A right cylinder and a cone have a common base and are also coaxial. The common base has a radius of 10 inches and the common axis is also 10 inches. Find the radius of the sphere whose volume equals the solid between the cylinder and the cone.

26. A right circular cone is inscribed in a sphere. The base of the cone is a great circle of the sphere. Find the ratio of the volume of the cone to the volume of the sphere.

27. Find the diameter of a 12-pound shot if one cubic inch of the metal weighs .28 pounds.

28. The radius of a sphere is 10 inches. Find the ratio of the volume of the sphere to the volume of a circumscribed equilateral cone.

29. How many spherical shot .15 inch in diameter can be made from a cylindrical bar of metal 2 inches in diameter and 1 foot long?

30. Find the diameter of a sphere whose volume equals the sum of the volumes of two spheres 3 inches and 5 inches in diameter.

31. Two spheres of radii 2 inches and 3 inches are melted and recast into a new sphere. Find the area of the new sphere.

32. The base of a spherical cone has an area of 12 square inches. If the radius of the sphere is 10 inches, find the volume of the spherical cone. (Use § 298.)

33. In a sphere of radius 16 inches a spherical wedge has a dihedral angle of 60°. Find the volume of the spherical wedge and the area of its lune.

34. Find the volume of a sphere circumscribed about a regular octahedron whose edge is 5 inches.

35. Find the area of a sphere in terms of its volume V.

36. The base of a spherical sector is a zone of altitude 3 inches. If the radius of the sphere is 8 inches, find the volume of the sector.

37. The base of a spherical sector is included between two parallel planes 3 inches apart. If the radius of the sphere is 10 inches, find the volume of the sector.

38. The area of the base of a spherical pyramid is $\frac{1}{5}$ the area of the sphere whose radius is 6 inches. Find the volume of the spherical pyramid.

39. What is the dihedral angle of a wedge that has $\frac{1}{10}$ the volume of the sphere?

40. A spherical sector is generated by revolving a 60° plane sector about one of its radii. If the radius of the sphere is 8 inches, find the volume of the spherical sector.

41. A sphere 4 inches in diameter is placed in an equilateral conical container. How much water must be poured into the container to cover the sphere?

42. Find the volume of a sphere circumscribed about a regular tetrahedron whose edge is 12 inches.

43. What is the ratio of the volumes of a cube and a sphere if they have equal areas?

44. The volumes of two spheres have the ratio of 64:125, and the sum of their radii is 18 inches. Find the radius of the smaller sphere.

45. The diameter of one sphere is twice that of another sphere; the sum of their volumes is 96π cubic inches. Find the area of the larger sphere.

46. A hemispherical bowl 12 inches in diameter is $\frac{2}{3}$ full of water. If the water is emptied into a cylinder 8 inches in diameter, how high will the water be?

47. Find the ratio of the volumes of two spheres that are circumscribed about and inscribed in the same cube.

48. Find the ratio of the volume of each sphere in Exercise 47 to the cube.

49. By what per cent must the radius of a sphere be increased so that the volume would be trebled?

50. An equilateral cone is inscribed in a sphere. What per cent of the volume of the sphere is outside the cone?

51. The inner diameter of a spherical shell is d and its thickness is t. Express the volume of the shell in terms of d and t.

52. The base of a right circular cone is the great circle of a hemisphere. Prove that if the volume of the cone equals the volume of the hemisphere, the lateral area of the cone is to the curved area of the hemisphere as $\sqrt{5}$ is to 2.

53. An isosceles triangle is inscribed in a semicircle and the figure is revolved about the diameter, which is 12 inches. Find the volume of the solid generated by the part of the figure between the triangle and the semicircle. (The diameter is one side of the triangle.)

54. An equilateral conical container 6 inches deep is filled with water. The largest sphere that can be completely immersed is placed in the container. How much water is left in the container?

55. The diameters of two hemispherical bowls are 6 inches and 9 inches. If a certain amount of water fills one-half the smaller bowl, what part of the larger bowl would it fill?

56. Find the diameter of a blackboard globe by the following method:

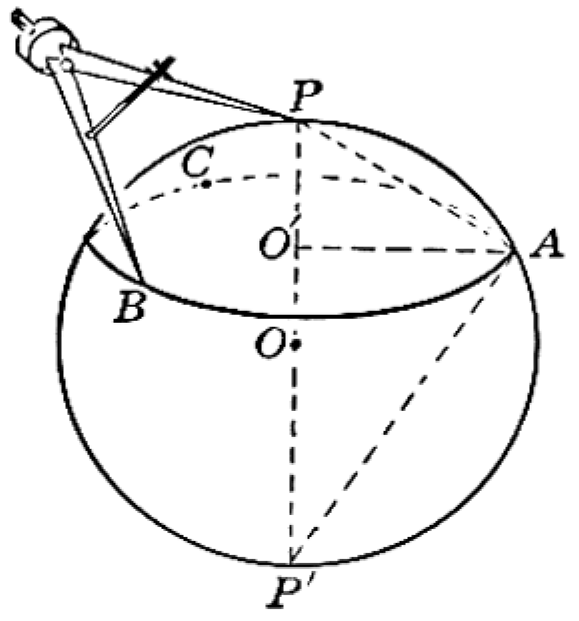


Fig. 1

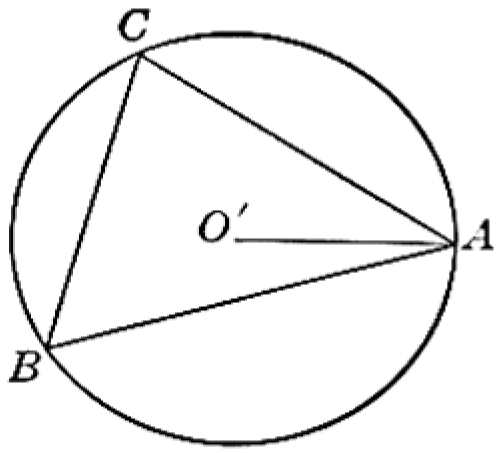


Fig. 2

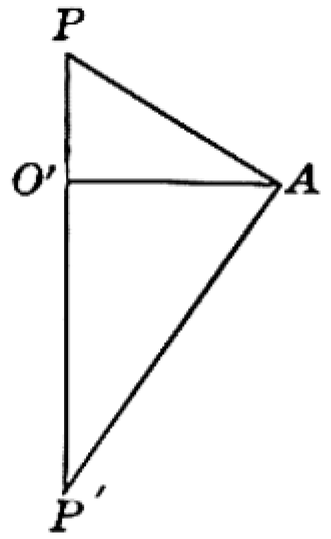


Fig. 3

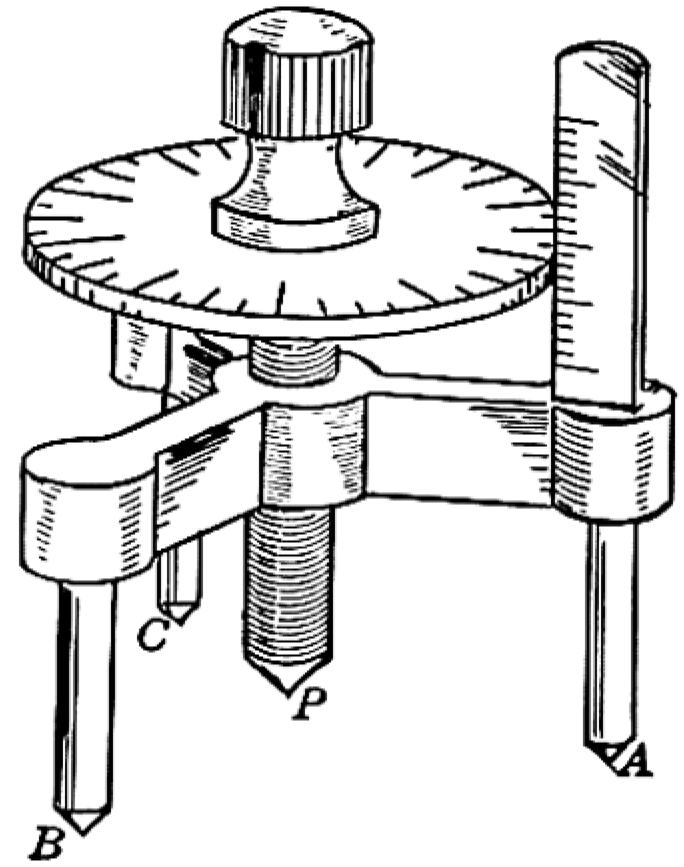
Fig. 1. With P as a pole and PA as a radius draw the $\odot ABC$ on the sphere.

Fig. 2. Using chords AB, BC, and CA from Fig. 1 as the sides, construct $\triangle ABC$ and circumscribe a circle about it.

Fig. 3. Construct this figure, using AP from Fig. 1, and $O'A$ from Fig. 2. The hypotenuse PP' is the required diameter.

SUPERIOR WORK

S 1. The instrument shown is called a *spherometer*. It is used to measure the radius of a sphere or the radius of curvature of a spherical surface such as a lens. The points A, B, and C, the fixed vertices of an equilateral triangle, rest on the sphere. The point P can be moved up or down by a micrometer screw which indicates the distance m from P to the plane ABC.



By referring to the diagram for Exercise 56, it will be seen that m is the distance $O'P$. $O'A$ is a fixed distance k for each instrument. It is the radius of the circle circumscribed about equilateral triangle ABC.

a. Derive a formula for the radius r of a sphere in terms of m and k.

b. In a certain spherometer $k = 2$ centimeters. Find the radius of curvature of a lens when $m = .15$ centimeter.

c. What is the meaning of a negative reading on the spherometer? Explain how a spherometer can be used to measure the thickness of a small object.

S 2. A solid sphere 12 inches in diameter sinks in water to a depth of 4 inches. Find the specific gravity of the material of the sphere. *Help:* The segment of the sphere immersed in the water displaces a volume of water whose weight equals the weight of the sphere.

$$\text{Specific gravity of sphere} = \frac{\text{weight of sphere}}{\text{weight of an equal volume of water}}.$$

S 3. A hollow metal sphere whose outer diameter is 8 inches just floats in water. If the specific gravity of the metal is 7, find the thickness of the metal.

S 4. In a sphere of radius 10 inches a spherical segment of one base has the radius of that base equal to 6 inches. Find the volume of the segment.

S 5. In a sphere of radius 13 inches a spherical segment of two bases has radii of 12 inches and 5 inches. Find the volume of the segment.

S 6. Using the formula in § 297, prove that the volume of a spherical segment of one base of radius r_1 and altitude h is given by the formula

$V = \frac{1}{6}\pi h(3r_1^2 + h^2)$. *Hint:* First show that $r = \dfrac{r_1^2 + h^2}{2h}$.

Note: By a bit of involved algebra, it may be shown that the volume of a spherical segment of two bases whose radii are r_1 and r_2 and whose altitude is h is given by the more general formula

$$V = \tfrac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2).$$

S 7. A sphere 12 inches in diameter is cut by a plane 2 inches from the center. Find the volume of each part.

S 8. Find the volume of a spherical segment if its altitude is 2 inches and the radii of its bases are 6 inches and 8 inches. Check your answer by using the formula in S 6.

S 9. A solid aluminum ball of 10-inch diameter has a 6-inch hole bored through the center. How many cubic inches of metal are left?

S 10. A cylindrical storage tank has an over-all length of 5 feet and a diameter of 18 inches. The ends of the tank are in the form of spherical segments. If the straight part of the tank is 4 feet long, find the capacity of the tank in gallons. Disregard the thickness of the metal.

S 11. A spherical steel shell whose outer diameter is 6 inches and whose thickness is $\frac{1}{2}$ inch is cut by a plane 2 inches from the center. Find the volume of the steel in each part of the shell.

S 12. Two parallel planes divide the diameter of a sphere to which they are perpendicular in the ratios 2:3:4. Find the ratios of the volumes of the three parts of the sphere.

Spherical Geometry

FIGURES ON A SPHERE

300. Spherical geometry is the study of figures on the surface of a sphere much as plane geometry is the study of figures in a plane surface. We shall presently see that there are distances, angles, and polygons on a sphere that are in some ways like and in other ways unlike such figures in a plane.

Spherical Distance

301. Of two circles, the one with the larger radius is said to have the less curvature. In fact, the curvature of a circle is the reciprocal of its radius. If two arcs terminate in the same two points, the arc with the lesser curvature (that is, the arc that is more nearly straight) is the shorter. A line segment has a curvature zero, or an infinite radius, and it is the shortest distance between two points.

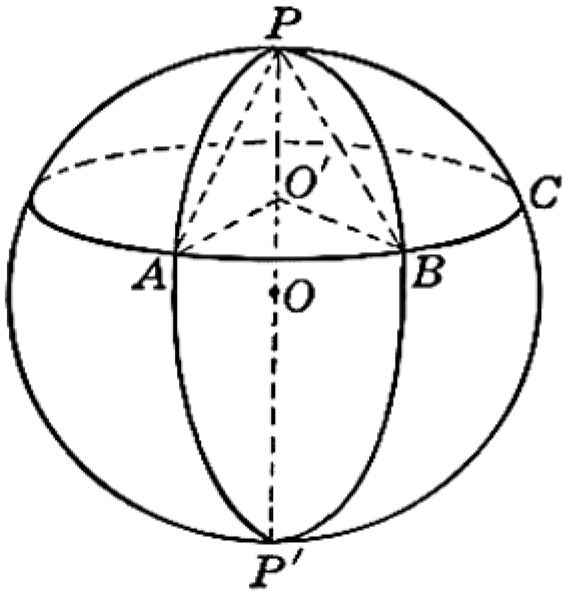
A straight line cannot be drawn on the surface of a sphere. On a sphere the arc with the smallest curvature, or the largest radius, is the arc of a great circle. Hence the shortest distance on a sphere between two points is the minor arc of a great circle between these points.

302. Postulate. *The shortest distance between two points on a sphere is the minor arc of a great circle through these points.*

303. Definition. *The spherical distance between two points on a sphere is the minor arc of a great circle through these points.*

The measure of a spherical distance may be in linear units or in degrees. Unless otherwise stated, the measure is in degrees.

304. THEOREM. All points on a circle of a sphere are the same spherical distance from either pole of the circle.



Given: P and P' poles of $\odot ABC$ on sphere O.

A and B any two pts. on $\odot ABC$.

PAP' and PBP' great circles.

Prove: $\overset{\frown}{PA} = \overset{\frown}{PB}$, $\overset{\frown}{P'A} = \overset{\frown}{P'B}$.

Cons.: Draw axis PP', intersecting plane ABC in O'.

Draw $O'A$, $O'B$, PA, and PB.

Proof: *Statements*	*Reasons*
1. $PP' \perp$ plane ABC.	1. Def. axis. (§ 249)
2. O' is the center of $\odot ABC$.	2. The axis of a $\odot$ of a sphere passes through the center of the $\odot$. (§ 250)
3. $PA = PB$.	3. If lines from pt. to plane cut = distances from foot of $\perp$ they are =. (§ 38)
4. $\odot PAP' = \odot PBP'$.	4. Great Ⓢ of same sphere are =.
5. $\overset{\frown}{PA} = \overset{\frown}{PB}$. Similarly, $\overset{\frown}{P'A} = \overset{\frown}{P'B}$.	5. In = Ⓢ, = chords have = arcs.

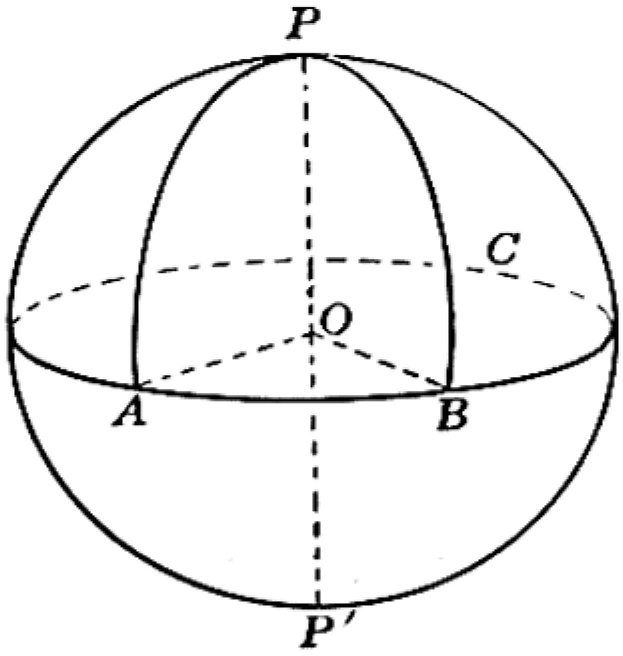
305. Definition. *The polar distance of a circle of a sphere is the spherical distance from any point on the circle to the nearer pole.*

306. Corollary. On the same sphere or on equal spheres equal circles have equal polar distances.

307. Definition. *A quadrant is one-fourth of a great circle.*

308. Corollary. The polar distance of a great circle is a quadrant.

309. THEOREM. If a point on a sphere is at a quadrant's distance from each of two points not the ends of a diameter, the point is a pole of the great circle passing through these points.



Given: A and B not ends of a diam.
ABC a great $\odot$ through A and B.
$\overset{\frown}{PA}$ and $\overset{\frown}{PB}$ quadrants.

Prove: P is a pole of $\odot ABC$.

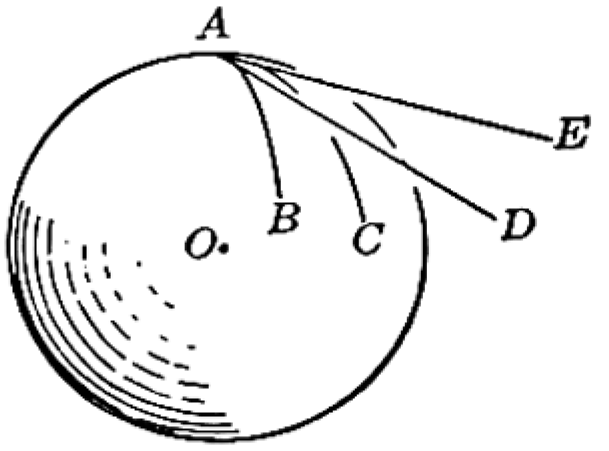
Cons.: Draw PO, OA, and OB.

Proof: *Statements*

1. $\angle POA$ and $\angle POB$ are rt. $\angle$s.
2. $PO \perp$ plane ABC.
3. POP' is axis of $\odot ABC$.
4. P is a pole of $\odot ABC$.

Supply the reason for each statement.

Spherical Angle

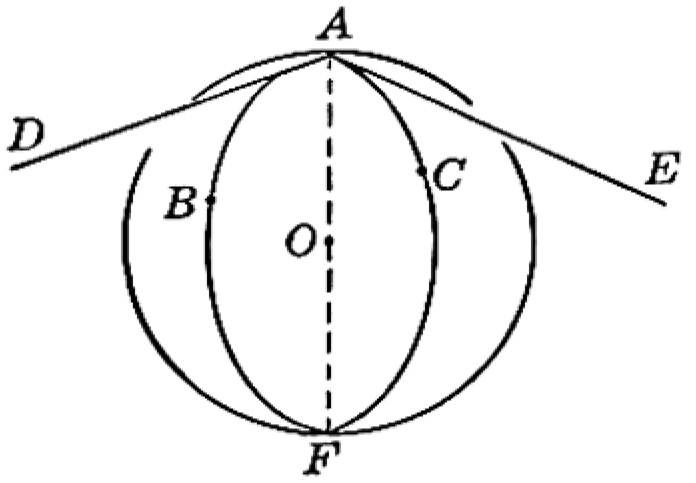


310. Definition. *A spherical angle is a figure on a sphere formed by two great-circle arcs from the same point. The arcs are called the sides of the angle; the common point is called the vertex.*

311. Definition. *The measure of a spherical angle is the measure of the angle between the tangents to the sides of the spherical angle at the vertex.*

On sphere O, AB and AC are great-circle arcs from A. $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ are the sides, and A is the vertex of spherical $\angle BAC$. AD is tangent to $\overset{\frown}{AB}$ at A, and AE is tangent to $\overset{\frown}{AC}$ at A. Spherical $\angle BAC \doteq \angle DAE$.

312. THEOREM. The measure of a spherical angle equals the measure of the dihedral angle formed by the planes of its sides.



GIVEN: Spherical $\angle BAC$.
Dihed. $\angle B{-}AF{-}C$ made by the planes of $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$.

PROVE: Spherical $\angle BAC \doteq$ dihed. $\angle B{-}AF{-}C$.

CONS.: Draw AD tangent to $\overset{\frown}{AB}$ at A, and AE tangent to $\overset{\frown}{AC}$ at A.

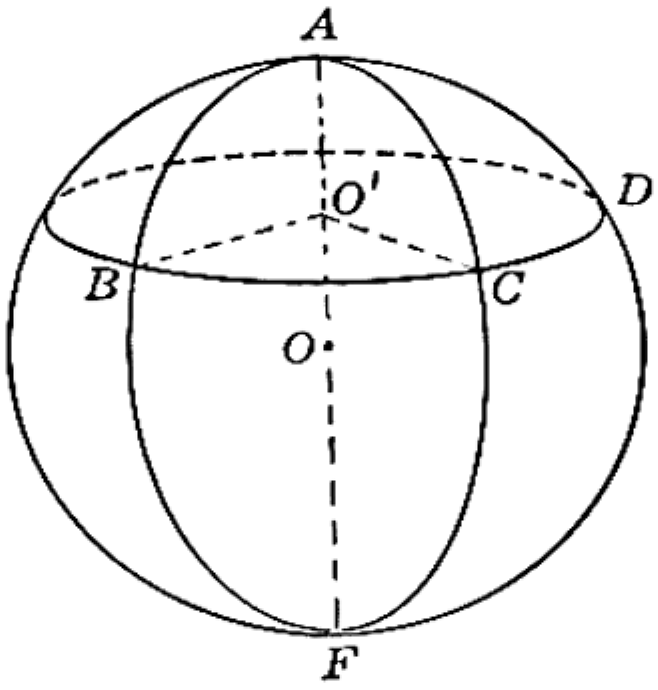
Supply the reason for each statement.

PROOF: *Statements*

1. Spherical $\angle BAC \doteq \angle DAE$.
2. In plane ABF, $AD \perp AF$ at A.
   In plane ACF, $AE \perp AF$ at A.
3. $\angle DAE$ is a plane $\angle$ of dihed. $\angle B{-}AF{-}C$.
4. Dihed. $\angle B{-}AF{-}C \doteq \angle DAE$.
5. Spherical $\angle BAC \doteq$ dihed. $\angle B{-}AF{-}C$.

What point connected with a circle in a plane corresponds to a pole of a circle on a sphere? What proposition in plane geometry corresponds to § 304? What proposition in plane geometry corresponds to the next theorem which you are about to prove?

313. THEOREM. The measure of a spherical angle equals the angular measure of the arc that it intercepts on a circle whose pole is the vertex of the spherical angle.



GIVEN: A, the vertex of spherical $\angle BAC$, is the pole of $\odot BCD$.

PROVE: Spherical $\angle BAC \doteq \widehat{BC}$.

CONS.: Draw axis AF of $\odot BCD$ intersecting plane BCD at O'. Draw $O'B$ and $O'C$.

Show that $\angle BO'C$ is the plane angle of dihedral $\angle B\text{–}AF\text{–}C$. Then use § 312.

Write out the proof.

314. Corollary. All the great-circle arcs drawn through the pole of a given great circle make right angles with the given circle.

315. Definitions. *Spherical angles are adjacent, vertical, supplementary, complementary, obtuse, or acute under the same conditions as plane angles. Two great-circle arcs are perpendicular if they form a spherical right angle.*

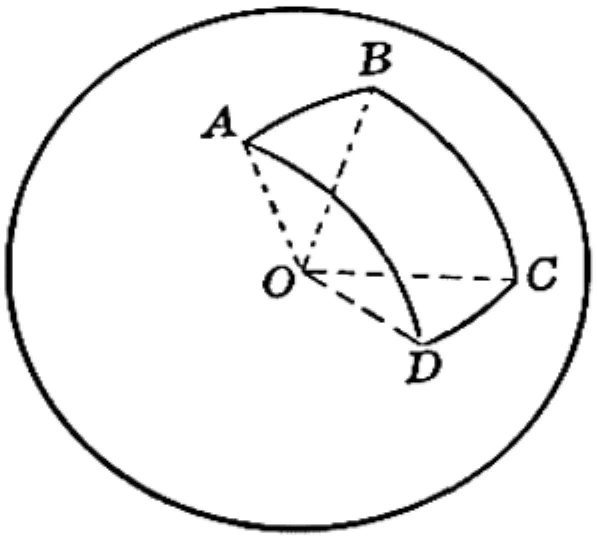
Spherical Polygon

316. Definition. *A spherical polygon is a closed figure on a sphere made by great-circle arcs.*

317. Corollary. If a polyhedral angle has its vertex at the center of a sphere, its intersection with the sphere is a spherical polygon.

We shall refer to a polyhedral angle and the spherical polygon that it forms as *corresponding figures.* A convex (§ 98) polyhedral angle forms a convex spherical polygon, the only kind we shall study.

In the figure, $ABCD$ is a spherical polygon; $O\text{–}ABCD$ is the corresponding polyhedral angle. When there is no doubt, we shall refer to the sides as AB, BC, CD, etc., instead of as $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$.

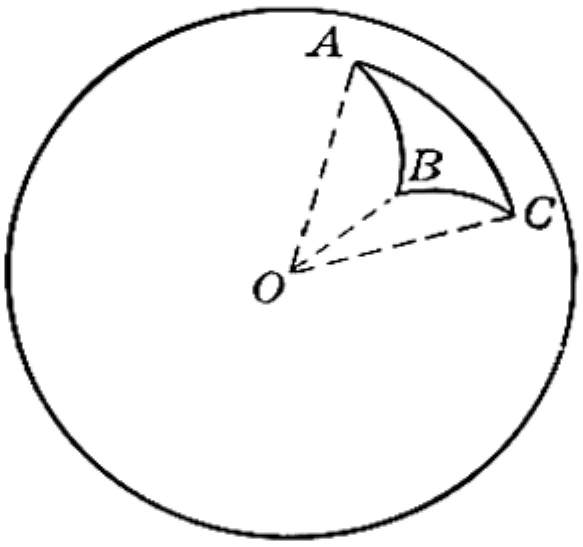


318. Corollary. The angular measures of the sides of a spherical polygon are the face angles of the corresponding polyhedral angle; the measures of the angles of a spherical polygon are the measures of the dihedral angles of the corresponding polyhedral angle.

319. Definitions. *A spherical triangle is a spherical polygon of three sides. A spherical triangle is isosceles, equilateral, equiangular, right, or obtuse under the same conditions as in plane geometry.*

320. Definitions. *The words median, altitude, bisector, and diagonal have the same relative meaning as in plane geometry.*

321. THEOREM. The sum of two sides of a spherical triangle is greater than the third side.



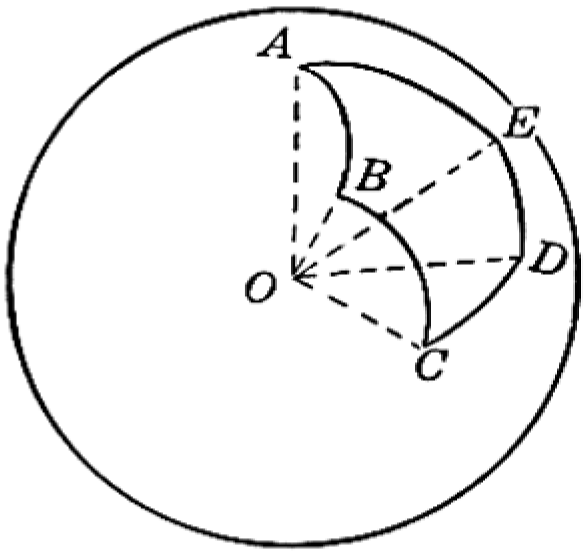
GIVEN: Spherical $\triangle ABC$.

PROVE: $AB + BC > AC$.

CONS.: Draw the corresponding trihedral $\angle O\text{–}ABC$.

PROOF: *Statements*	*Reasons*
1. $AB = \angle AOB$. $BC = \angle BOC$. $AC = \angle AOC$.	1. The angular measure of an arc is its central angle. (§ 517)
2. $\angle AOB + \angle BOC > \angle AOC$.	2. Sum of 2 face ∠s of a trihed. ∠ > third face ∠. (§ 99)
3. $AB + BC > AC$.	3. Subs. Ax.

322. THEOREM. The sum of the sides of a spherical polygon is less than 360°.



GIVEN: Spherical polygon $ABCD \cdots$.

PROVE: $AB + BC + CD + \cdots < 360°$.

CONS.: Draw the corresponding polyhedral $\angle O\text{–}ABCD \cdots$.

Write out the proof. (Use § 100.)

EXERCISES

1. On a sphere 12 inches in diameter find the polar distance of a circle 6 inches in diameter.

2. On a sphere of 8-inch radius the polar distance of a circle is 60°. Find the area of the smaller zone formed by this circle.

3. Prove that each side of a spherical triangle is less than two quadrants.

4. Is it possible for a spherical triangle to have the following sides?

   a. 40°, 100°, 150°
   b. 40°, 100°, 130°
   c. 40°, 100°, 60°
   d. 90°, 90°, 90°
   e. 160°, 150°, 30°
   f. 120°, 140°, 110°

5. In each of the following, two sides of a spherical triangle are given. Determine between what limits the third side must be.

   a. 90°, 90°
   b. 100°, 50°
   c. 100°, 80°
   d. 150°, 110°
   e. 160°, 20°
   f. 170°, 170°

6. Determine whether or not the following can be the sides of a spherical polygon. (Refer to Exercise 4, page 64.)

   a. 80°, 60°, 90°, 70°, 50°
   b. 70°, 120°, 110°, 60°
   c. 60°, 30°, 150°, 50°
   d. 40°, 85°, 30°, 20°

7. Prove that vertical spherical angles are equal.

8. Prove that if two angles of a spherical triangle are right angles, the sides opposite are quadrants.

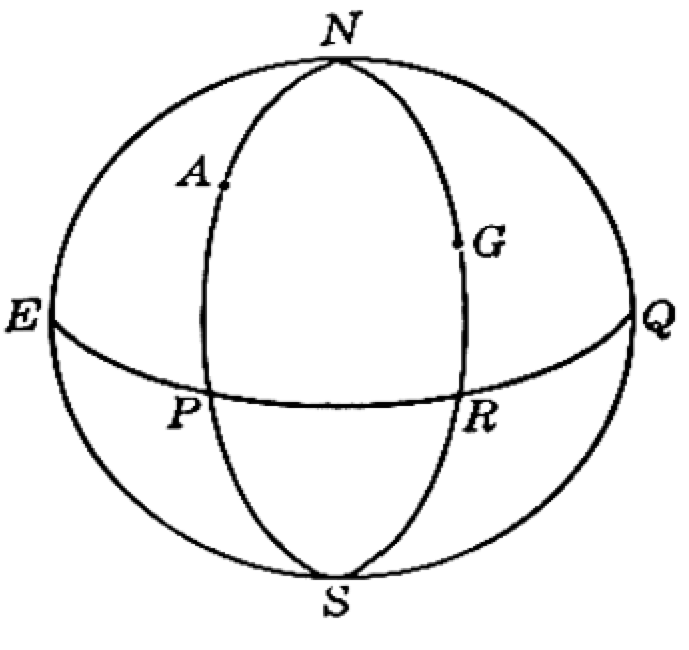
9. Prove:

   a. If a spherical triangle has two right angles, all its angles are measured by the opposite sides.
   b. If a spherical triangle has three right angles, each side is a quadrant.

10. The sides of a spherical triangle are 30°, 70°, and 80°. If the radius of the sphere is 10 feet, find the number of feet in the perimeter of the spherical triangle.

11. Prove that the meridians of the earth (page 148, Exercise 24) are perpendicular to the equator.

12. In the figure N and S are the North and South poles of the earth, EQ is the equator, G is Greenwich, England, A is any other point on the earth, and circles $NGRS$ and $NAPS$ are meridians.



   a. The *latitude* of a point is the part of the meridian between that point and the equator. What is the latitude of G? of A? Are these latitudes north or south of the equator?

b. The meridian through Greenwich, England, (G), is called the *prime meridian.* The longitude of a point is the part of the circle of latitude between the meridian through that point and the prime meridian. What is the longitude of point A? Is it east or west of Greenwich?

c. The longitudes of two points are 40° E and 15° E. What is the difference in longitude (abbreviated DL_o) between these points?

d. Remembering that the earth rotates through 360° longitude in 24 hours, what is the difference in time between the two points in (c)?

e. At the North Pole, which direction is east? north? south?

13. The length of one minute ($\frac{1}{60}$ of a degree) on the equator is called a nautical mile. If the diameter of the equator is taken as 7930 land miles, by what per cent is a nautical mile greater than a land mile?

14. Two ports on the same meridian have latitudes of 15° N and 27° N. Find the distance between them in nautical miles. (See Exercise 13.)

15. The angle between two meridians is 15°. How many nautical miles are on the equator between these meridians?

16. Two cities on the 30th parallel have a difference in longitude of 40°. Find the distance in land miles along the equator between their meridians.

17. Prove that two circles on a sphere are equal if they have equal polar distances.

18. On a sphere of radius 4 inches two sides of a spherical triangle are each 2π inches long and the included angle is 50°. Find the length of the third side.

19. Points A, B, and C are located on the earth as follows: A: 80° W, 40° N. B: 170° W, 0°. C: 80° W, 0°. Find the angles of the spherical triangle ABC.

20. Prove that the part of the earth between the latitudes of 30° N and 30° S is one-half the whole surface of the earth.

SUPERIOR WORK

S 1. The circles of latitude at $23\frac{1}{2}$° N and $23\frac{1}{2}$° S are called the *Tropics of Cancer* and *Capricorn* respectively. The zone between these circles is called the *Torrid Zone.* Find the ratio of the area of the Torrid Zone to the area of the earth.

S 2. The circles of latitude at $67\frac{1}{2}$° N and $67\frac{1}{2}$° S are called the *Arctic* and *Antarctic Circles* respectively. The zone between the Tropic of Cancer and the Arctic Circle is called the *North Temperate Zone.* The zone between the Tropic of Capricorn and the Antarctic Circle is called the *South Temperate Zone.* What part of the earth's surface is each temperate zone?

S 3. The *North Frigid Zone* lies between the Arctic Circle and the North Pole. Define the *South Frigid Zone.* What part of the earth's surface is each frigid zone?

S 4. The *departure* between two meridians is the length of the arc of the parallel of latitude subtended by these meridians.

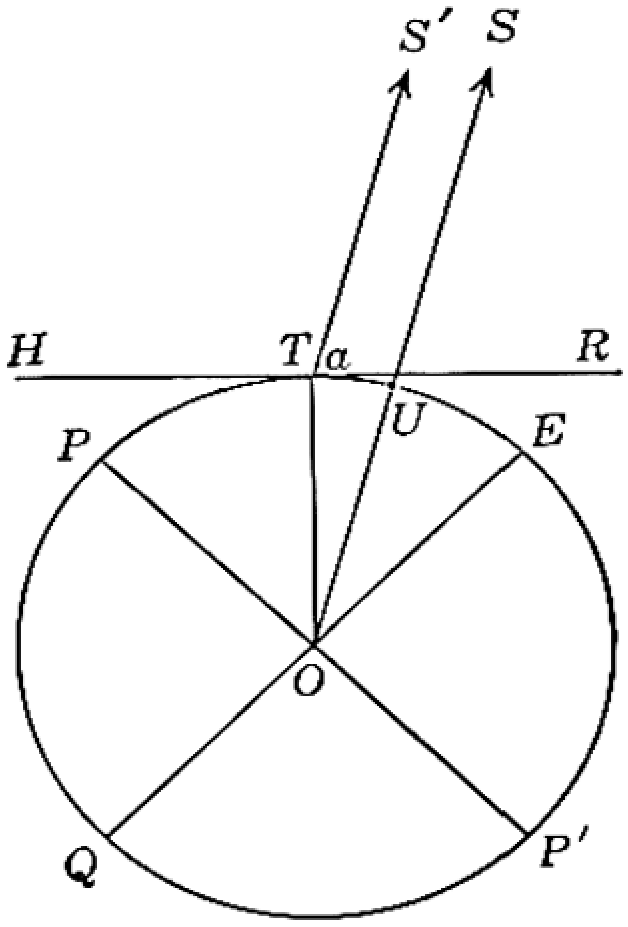
Prove that the departure between two meridians equals the length of the equator between these meridians times the cosine of the latitude of the departure.

S 5. In the figure, T represents a point on the surface of the earth, HR is the horizon at T, PP' is the axis of the earth, and EQ is the equator.

OS is a line from the center of the earth to a certain star. The point U is directly "under this star." The arc EU, which is the latitude of U, is called the *declination* of the star.

At T the angle of elevation of the star is $\angle a$. This is called the *altitude* of the star.

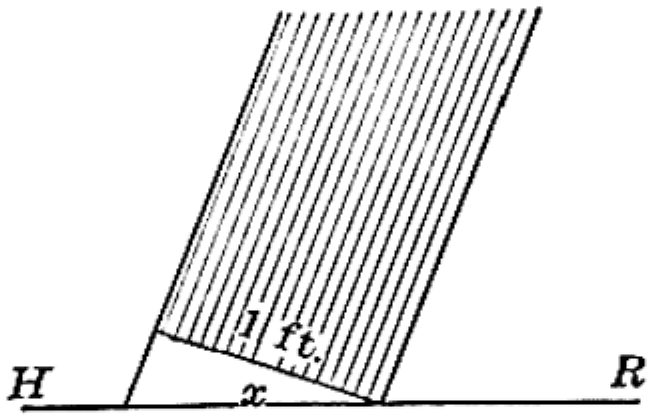
Prove that at T the altitude of the star equals the colatitude (complement of the latitude) of T plus the declination of the star; that is, $a = (90^\circ - l) + d$.



S 6. a. On June 21 the declination of the sun is $23\frac{1}{2}^\circ$ N. Find the altitude of the sun at its highest point on that day as observed in New York, where the latitude is $40\frac{1}{2}^\circ$. (Use Exercise S 5.)

b. On December 21 the declination of the sun is $23\frac{1}{2}^\circ$ S, or $-23\frac{1}{2}^\circ$ N. Find the altitude of the sun at its highest point on that day as seen in New York.

S 7. The figure represents a rectangular beam of light from the sun 1 foot wide. Since this beam strikes the horizontal surface obliquely, it covers a rectangle on the ground whose width is x feet.



Using your answers to S 6, find the values for x at New York on December 21 and June 21. The ratio between these two answers compares the energy delivered by the sun at high noon on these two dates. Find this ratio.

S 8. a. The North Star, Polaris, is in a line with the axis of the earth. What is its declination?

b. Prove that at any point in the Northern Hemisphere, the altitude of Polaris equals the latitude of that point.

S 9. A navigator finds that the altitude of a star is 60°. Find his spherical distance from the point directly below the star. (Refer to the diagram in S 5.) In general, what is the relation between this spherical distance and the altitude?

Note: The navigator uses this principle in locating his position. The position of the point under the star (the substellar position) is found in astronomical almanacs for any given time. Using this position as the pole and the computed spherical distance as the polar distance, the navigator draws a circle. His own position is somewhere on this circle. He then repeats the process with another star. His position must then be one of the two points in which the circles intersect. Usually these two points are so far apart that there is no question about which to choose.

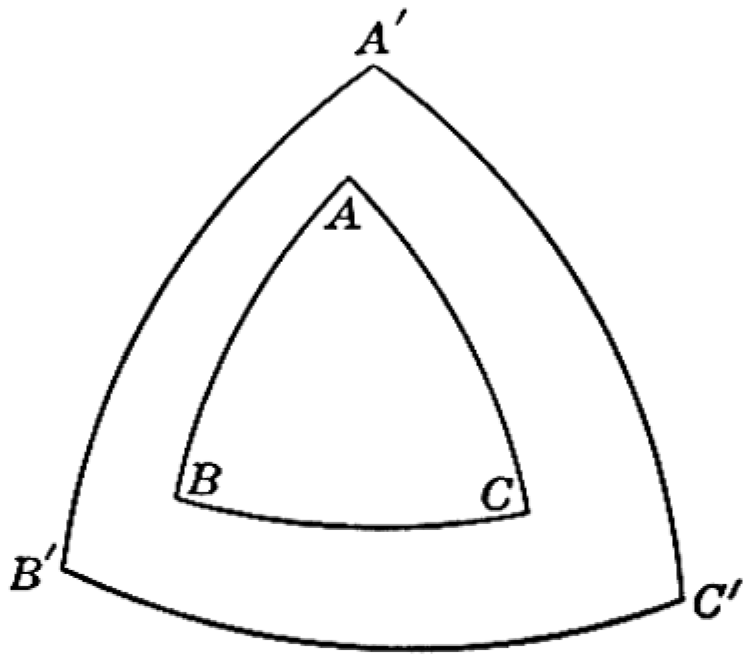
S 10. Two circles on a sphere have a common pole. Their polar distances are 60° and 45° respectively. If the radius of the sphere is 6 inches, find the volume of the spherical segment between the planes of the two circles.

POLAR TRIANGLES

Given a spherical triangle, we can use the vertices as poles and draw the sides of another spherical triangle. The new triangle is called the polar triangle of the original triangle.

323. Definition. *If the vertices of one spherical triangle are the poles of the sides of another spherical triangle, the second triangle is the polar triangle of the first.*

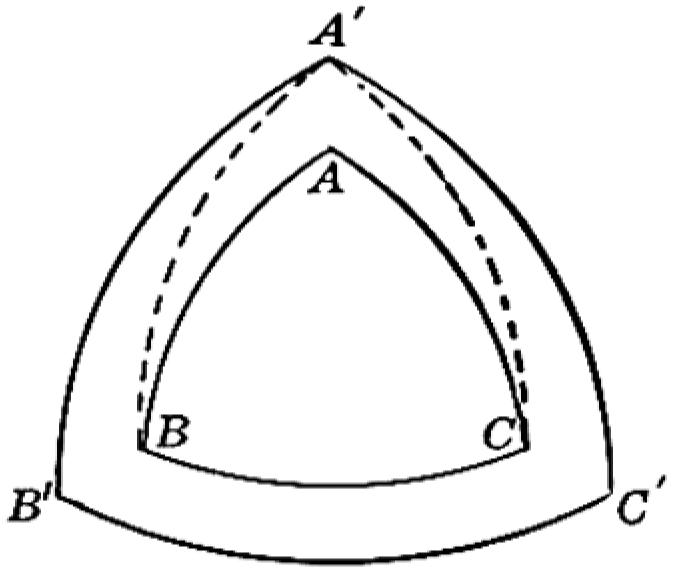
In the figure, A is the pole of $B'C'$, B is the pole of $A'C'$, and C is the pole of $A'B'$. $\triangle A'B'C'$ is the polar triangle of ABC.



Note: The arcs $A'B'$, $B'C'$, and $C'A'$, if extended, form eight spherical triangles. However, the polar triangle is the one where AA', BB', and CC' are each less than a quadrant. In other words, the polar triangle is the one which has its vertices nearest to the original triangle.

Our next theorem will show that if T' is the polar triangle of T, then T is also the polar triangle of T'. In other words, polar triangles is a reciprocal relationship. Name some other relationships like this and some that are not like this.

324. THEOREM. If one spherical triangle is the polar triangle of another, the second triangle is the polar of the first.



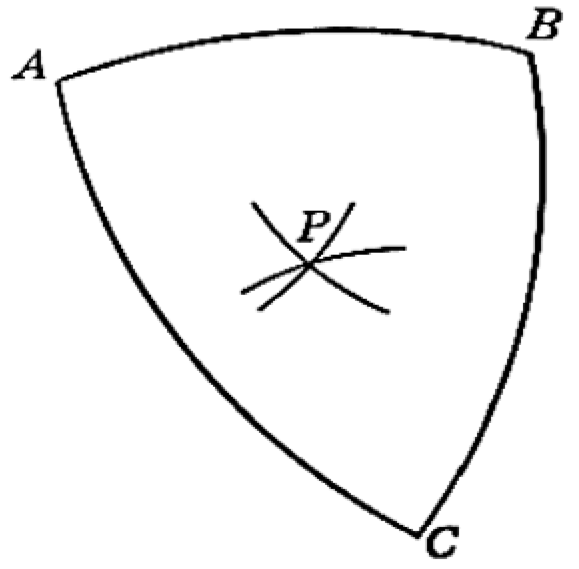
GIVEN: $\triangle A'B'C'$ the polar $\triangle$ of ABC.

PROVE: $\triangle ABC$ the polar $\triangle$ of $A'B'C'$.

PROOF:

Statements	*Reasons*
1. B is the pole of $A'C'$.	1. Def. polar $\triangle$.
2. $\widehat{A'B} = 1$ quadrant.	2. The polar distance of a great ⊙ is a quadrant. (§ 308)
3. C is the pole of $A'B'$.	3. Def. polar $\triangle$.
4. $\widehat{A'C} = 1$ quadrant.	4. Same as 2.
5. A' is the pole of BC.	5. If a pt. is a quadrant's distance from each of 2 pts., not ends of diam., the pt. is the pole of a great ⊙ through these pts. (§ 309)
Similarly, B' is the pole of AC, and C' is the pole of AB.	
6. $\triangle ABC$ is the polar $\triangle$ of $A'B'C'$.	6. Def. polar $\triangle$.

On a blackboard globe draw a spherical triangle in which one side is more than a quadrant. Then draw its polar triangle. Note that the two triangles overlap. For simplicity we shall draw our polar triangles so that they do not overlap.



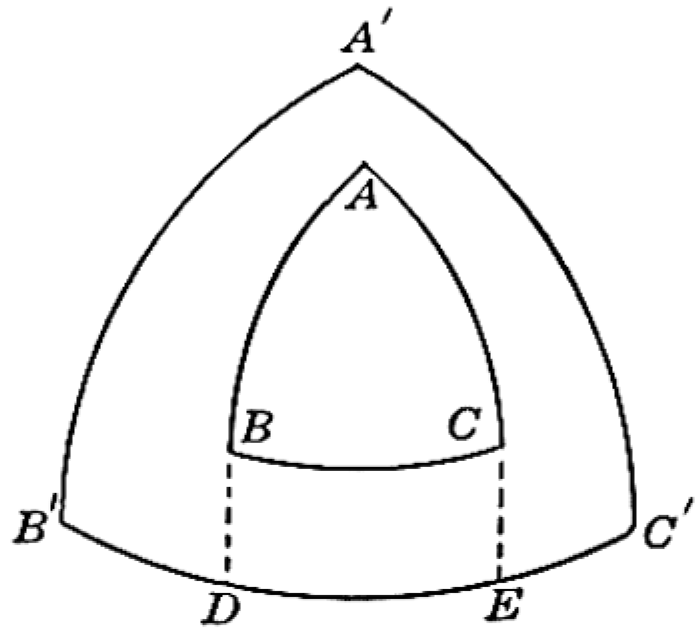
Draw a spherical triangle in which each side is a quadrant. Now draw its polar triangle. What do you observe?

Is it possible for a spherical triangle ABC not to have a polar triangle? The only way this could happen is for the three great-circle arcs, drawn with A, B, and C as poles, to meet in one point P. In that case A, B, and C would each be a quadrant's distance from P. They would then be on one great circle (§ 309).

By definition of a spherical triangle, therefore, ABC would not be a spherical triangle.

325. Corollary. Every spherical triangle has a polar triangle.

326. THEOREM. In two polar triangles, each angle of one triangle is the supplement of the opposite side of the other.



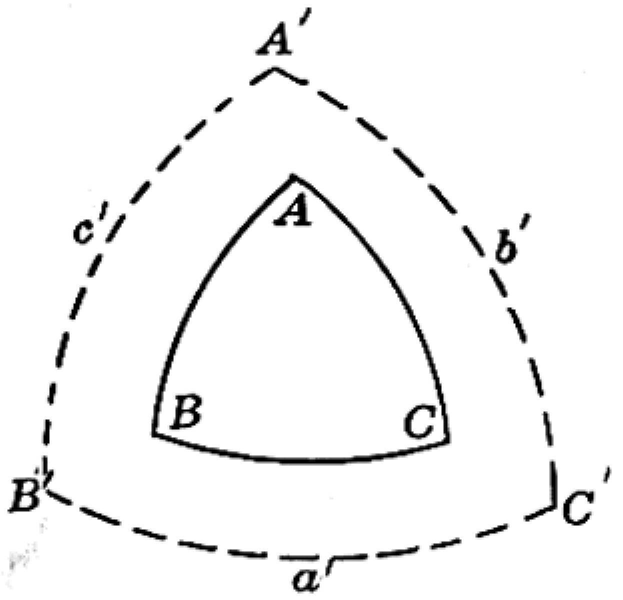
GIVEN: Polar $\triangle\!\triangle\, ABC$ and $A'B'C'$.

PROVE: $\angle A + B'C' = 180°$.

CONS.: Prolong AB and AC to meet $B'C'$ in D and E.

PROOF: *Statements*	*Reasons*
1. B' is the pole of AE, C' is the pole of AD.	1. Def. polar △.
2. $B'D + DE = 90°$. $DE + EC' = 90°$.	2. The polar distance of a great ⊙ is a quadrant. (§ 308)
3. $DE + (B'D + DE + EC') = 180°$.	3. Add. Ax.
4. $DE = \angle A$.	4. Spherical ∠ = arc intercepted on ⊙ whose pole is vertex of spherical ∠. (§ 313)
5. $B'D + DE + EC' = B'C'$.	5. Identity.
6. $\angle A + B'C' = 180°$.	6. Subs. Ax. (In Eq. 3)

327. THEOREM. The sum of the angles of a spherical triangle is greater than 180° but less than 540°.



GIVEN: Spherical $\triangle ABC$.

PROVE: $180° < \angle A + \angle B + \angle C < 540°$.

CONS.: Draw $A'B'C'$, the polar triangle of ABC.

Proof:

Statements	*Reasons*
1. $\angle A + a' = 180^\circ$. $\angle B + b' = 180^\circ$. $\angle C + c' = 180^\circ$.	1. In 2 polar △s, each ∠ of one is sup. of opp. side of the other. (§ 326)
2. $\angle A + \angle B + \angle C + a' + b' + c' = 540^\circ$.	2. Add. Ax.
3. $a' + b' + c' < 360^\circ$.	3. Sum of sides of spherical polygon < 360°. (§ 322)
4. $\angle A + \angle B + \angle C > 180^\circ$.	4. If ≠s are subtracted from =s, remainders ≠ in opp. order. (§ 444)
5. $a' + b' + c' > 0$.	5. Every spherical △ has a polar △. (§ 325)
6. $\angle A + \angle B + \angle C < 540^\circ$.	6. Same as 4. (Eq. 2 − Ineq. 5)

328. Corollary. A spherical triangle may have one, two, or three right angles; one, two, or three obtuse angles.

329. Definitions. *A right spherical triangle is one that has at least one right angle. A birectangular spherical triangle is one that has two right angles; a trirectangular spherical triangle is one that has three right angles. An oblique spherical triangle is one that has no right angle.*

330. Corollary. The sum of the dihedral angles of a trihedral angle is greater than 180° but less than 540°.

Note that we arrived at this fact through dynamic analysis in Exercise 8, page 64.

EXERCISES

1. a. If one of two polar triangles is completely inside the other, what can you say about their sides?
   b. Draw a spherical triangle with one right angle and then draw its polar triangle.
   c. Draw two polar triangles so that the sides of one are 50°, 90°, and 90°.
2. Prove that a trirectangular spherical triangle is its own polar triangle.
3. The sides of a spherical triangle are 40°, 70°, and 100°. Find the angles of its polar triangle.
4. The angles of a spherical triangle are 130°, 50°, and 70°. Find the sides of its polar triangle.
5. In a birectangular spherical triangle the third angle is 120°. Find the perimeter of the polar triangle in feet if the radius of the sphere is 10 feet.

6. If two angles of a spherical triangle are 50° and 90°, between what limits is the third angle?

7. Prove that if one of two polar triangles is birectangular, the other is also birectangular.

8. One side of a spherical triangle is 20°. If its opposite vertex is also its pole, find the sum of the angles of the triangle.

9. Is it possible for the following to be the angles of a spherical triangle? (Check the sides of the polar triangle.)

   a. 130°, 130°, 70°
   b. 110°, 100°, 20°
   c. 110°, 100°, 40°
   d. 70°, 70°, 30°
   e. 150°, 90°, 70°
   f. 120°, 110°, 60°

10. Prove that an exterior angle of a spherical triangle is less than the sum of the two remote interior angles.

11. Prove that the sum of the angles of a spherical polygon of n sides is greater than $(n - 2)$ straight angles but less than n straight angles.

12. If A, B, and C are the angles of a spherical triangle, prove that

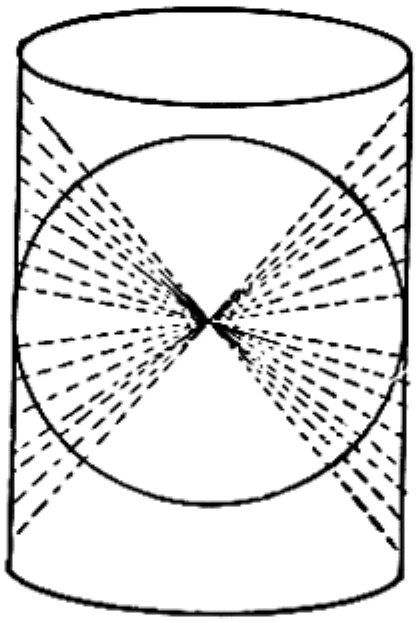
$$A + B < 180° + C.$$

SUPERIOR WORK

Map Making

The problems that follow concern the three principal methods of projecting the surface of the earth on a plane. All of them, of course, result in a certain amount of distortion.

S 1. The Mercator Projection. In this method a cylindrical surface is drawn tangent to the earth at the equator. Lines from the center of the earth project the points of the earth on the cylindrical surface. This is then rolled out into a plane.



a. In what form do the meridians appear on a Mercator projection? Are they equally spaced?

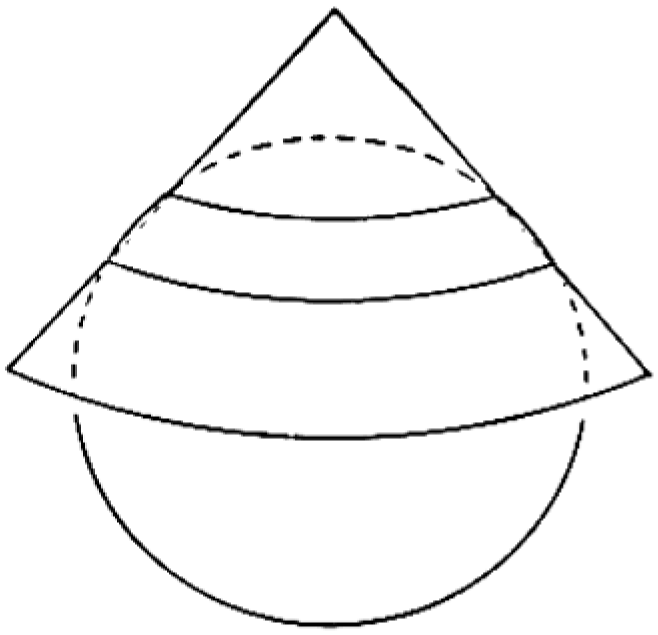
b. In what form are the circles of latitude? Are they equally spaced?

c. Find the ratio of the Mercator area to the actual area between the latitudes of 0° and 30°.

d. Find the ratio of the Mercator area to the actual area between 30° and 60° latitude.

e. Where is the greatest distortion on this map? the least?

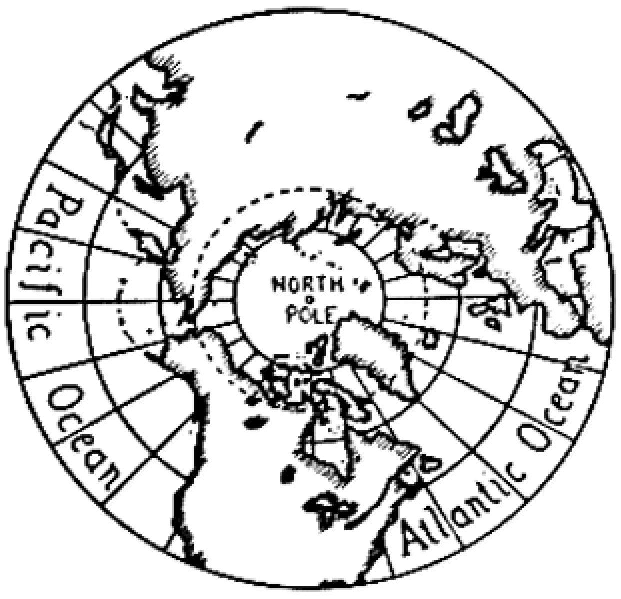
Note: In the areas of greatest distortion the parallels of latitude are moved closer together to overcome, in part, the distortion.

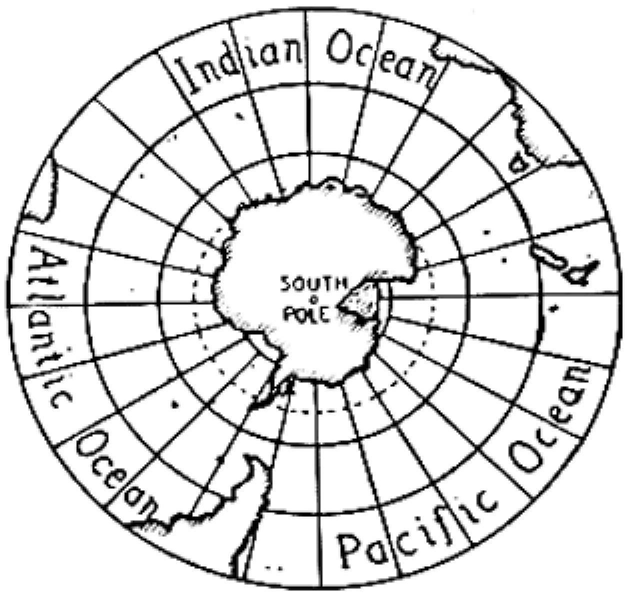


S 2. The Lambert Projection. Here lines are drawn from the center of the earth projecting points of the earth on a conical surface. The vertex of the cone is on the prolongation of the earth's axis. For maps of the United States, the cone passes through the circles of latitude 33° N and 45° N. The conical surface is then rolled out into a plane.

- **a.** In what form are the meridians in the Lambert projection? Are they equally spaced?
- **b.** In what form are the circles of latitude? Are they equally spaced?
- **c.** Find the ratio of the Lambert area to the actual area between 33° and 45° latitude.
- **d.** Where is the greatest distortion on this map? the least?

S 3. Plane Projection. In this map the points of the earth are projected directly on a plane which is tangent to the earth. In the North Polar projection, the projecting lines are drawn from the center of the earth, and the plane is tangent to the earth at the North Pole.





- **a.** In what form are the meridians in the polar projection? Are they equally spaced?
- **b.** In what form are the circles of latitude? Are they equally spaced?
- **c.** What does the North Polar projection show about the relative amounts of land and water? about the relation between North America and Asia? About what part of the Pole does Russia surround? Which direction is east? west? north? south?
- **d.** About what part of the South Polar projection map is water? Name the three oceans that appear as one ocean. Where is the greatest distortion on this map?
- **e.** Find the ratio of the polar projection area to the actual area between 60° and 90° latitude.
- **f.** The flag of the United Nations uses a modified polar projection to show parts of the Southern Hemisphere. Study that emblem and tell how it differs from the polar projection. Where is the greatest distortion?

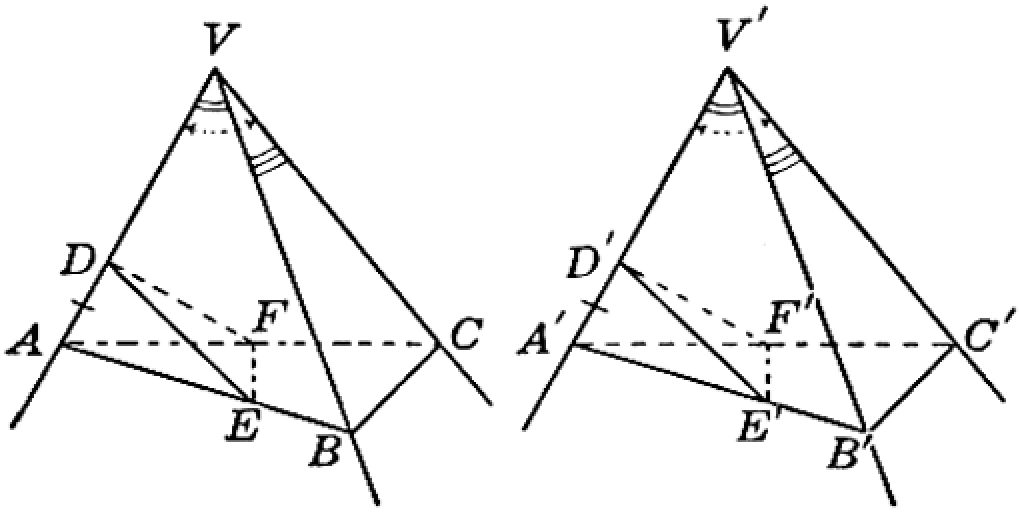
CONGRUENT SPHERICAL TRIANGLES

The definition of congruent and symmetric solids in § 109 applies to spherical triangles.

331. Definition. *On the same sphere or on equal spheres, if two spherical triangles have the parts of one equal respectively to the corresponding parts of the other, they are congruent or symmetric according as the parts are arranged in the same order or in the opposite order.*

332. Corollary. On the same sphere or on equal spheres spherical triangles are congruent or symmetric according as their corresponding trihedral angles are congruent or symmetric; and conversely.

333. THEOREM. If two trihedral angles have the three face angles of one equal respectively to the three face angles of the other, they are either congruent or symmetric. (f.f.f.)



GIVEN: In trihed. $\measuredangle V$ and V', $\angle AVB = \angle A'V'B'$, $\angle BVC = \angle B'V'C'$, $\angle CVA = \angle C'V'A'$.

PROVE: Trihed. $\angle V \cong$ trihed. $\angle V'$.

CONS.: Lay off $VA = VB = VC = V'A' = V'B' = V'C'$.
Draw AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$. Lay off $AD = A'D'$.
In plane AVB draw $DE \perp AV$ meeting AB at E.
In plane AVC draw $DF \perp AV$ meeting AC at F. Draw EF.
In like manner draw $D'E'$, $D'F'$, and $E'F'$.

PROOF: *Statements*	*Reasons*
1. $\triangle AVB \cong \triangle A'V'B'$. $\triangle BVC \cong \triangle B'V'C'$. $\triangle CVA \cong \triangle C'V'A'$.	1. s.a.s.
2. $\angle DAE = \angle D'A'E'$, $\angle DAF = \angle D'A'F'$, $AB = A'B'$, $BC = B'C'$, $CA = C'A'$.	2. Def. $\cong \triangle$.

3. $\triangle ABC \cong \triangle A'B'C'$.	3. s.s.s.
4. $\angle EAF = \angle E'A'F'$.	4. Def. $\cong$ ◬.
5. $\triangle ADE \cong \triangle A'D'E'$, $\triangle ADF \cong \triangle A'D'F'$.	5. a.s.a.
6. $AE = A'E'$, $DE = D'E'$, $AF = A'F'$, $DF = D'F'$.	6. Def. $\cong$ ◬.
7. $\triangle AEF \cong \triangle A'E'F'$.	7. s.a.s.
8. $EF = E'F'$.	8. Def. $\cong$ ◬.
9. $\triangle DEF \cong \triangle D'E'F'$.	9. s.s.s.
10. $\angle EDF = \angle E'D'F'$.	10. Def. $\cong$ ◬.
11. Dihed. $\angle VA$ = dihed. $\angle V'A'$.	11. Dihed. ∡ are = if their plane ∡ are =. (§ 78)

Similarly,

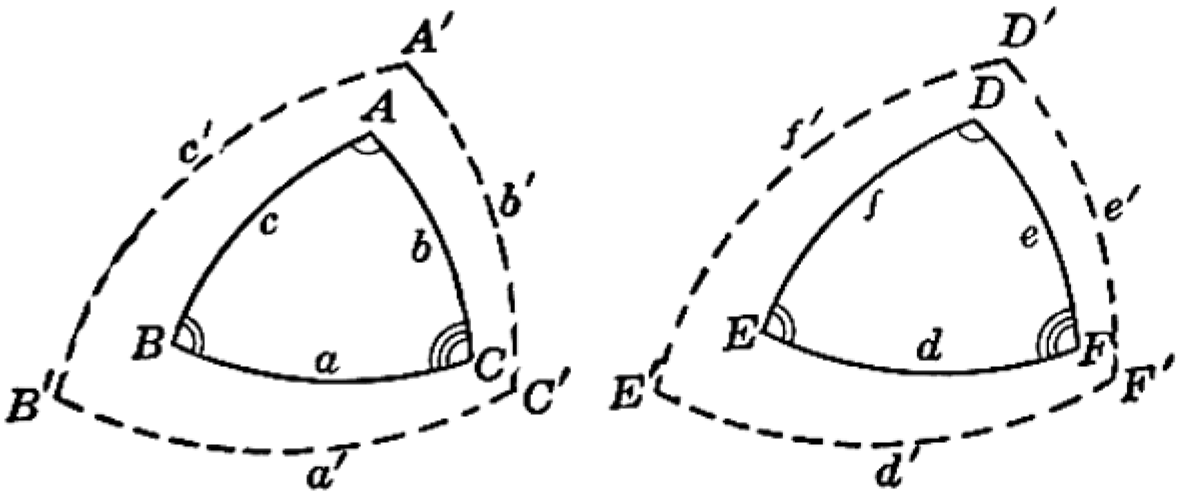
Dihed. $\angle VB$ = dihed. $\angle V'B'$.
Dihed. $\angle VC$ = dihed. $\angle V'C'$.

12. Trihed. $\angle V \cong$ trihed. $\angle V'$.	12. Def. $\cong$ solids. (§ 109)

Note: If their parts were arranged in opposite orders, the trihedral angles would be symmetric.

334. Corollary. On the same sphere or on equal spheres, if two spherical triangles have the three sides of one equal respectively to the three sides of the other, they are either congruent or symmetric. (s.s.s.)

335. THEOREM. On the same sphere or on equal spheres, if two spherical triangles have the three angles of one equal respectively to the three angles of the other, they are either congruent or symmetric. (a.a.a.)



Given: Spherical ◬ ABC and DEF on equal spheres.
$\angle A = \angle D$. $\angle B = \angle E$. $\angle C = \angle F$.

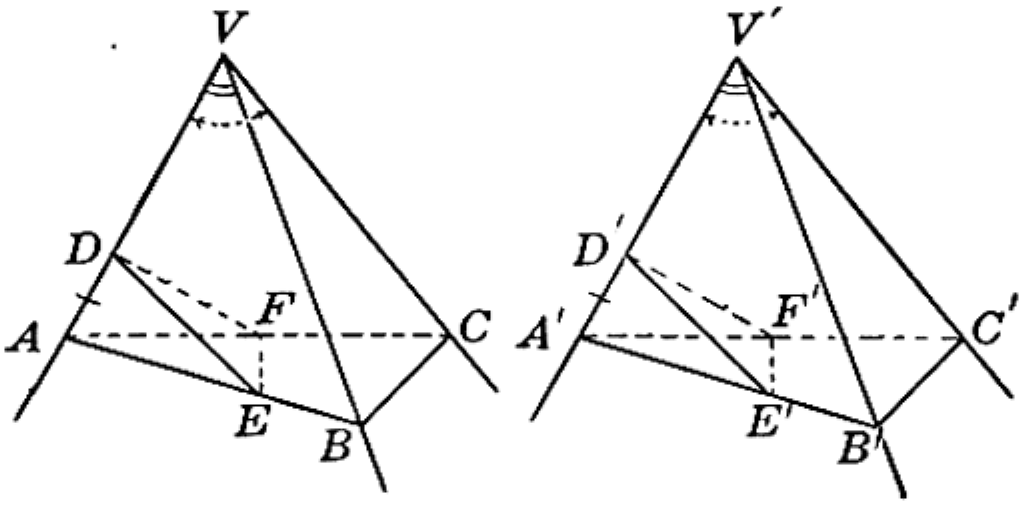
Prove: $\triangle ABC \cong \triangle DEF$.

Cons.: Draw the polar ◬ $A'B'C'$ and $D'E'F'$.

Proof: *Statements*	*Reasons*
1. $\angle A + a' = 180°$, $\angle D + d' = 180°$.	1. In 2 polar $\triangle$s, each $\angle$ of one is sup. of opp. side of the other. (§ 326)
2. $\angle A = \angle D$.	2. Given.
3. $a' = d'$. Similarly, $b' = e'$, $c' = f'$.	3. Sups. of = $\angle$s are =.
4. $\triangle A'B'C' \cong \triangle D'E'F'$.	4. s.s.s. (§ 334)
5. $\angle A' = \angle D'$.	5. Def. $\cong$ spherical $\triangle$s.
6. $\angle A' + a = 180°$. $\angle D' + d = 180°$.	6. Same as 1.
7. $a = d$. Similarly, $b = e$, $c = f$,	7. Same as 3.
8. $\triangle ABC \cong \triangle DEF$.	8. Def. $\cong$ spherical $\triangle$s.

336. Corollary. If two trihedral angles have the three dihedral angles of one equal respectively to the three dihedral angles of the other, they are either congruent or symmetric. (a.a.a.)

337. THEOREM. If two trihedral angles have two face angles and the included dihedral angle of one equal respectively to the corresponding parts of the other, they are either congruent or symmetric. (f.a.f.)



Given: In trihed. $\angle$s V and V', $\angle AVB = \angle A'V'B'$, $\angle AVC = \angle A'V'C'$, dihed. $\angle VA$ = dihed. $\angle V'A'$.

Prove: Trihed. $\angle V \cong$ trihed. $\angle V'$.

Cons.: Lay off $VA = VB = VC = V'A' = V'B' = V'C'$.
Draw AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$. Lay off $AD = A'D'$.
In plane AVB draw $DE \perp AV$ meeting AB at E.
In plane AVC draw $DF \perp AV$ meeting AC at F. Draw EF.
In like manner, draw $D'E'$, $D'F'$, and $E'F'$.

PROOF:

Statements	*Reasons*
1. $\triangle AVB \cong \triangle A'V'B'$, $\triangle AVC \cong \triangle A'V'C'$.	1. s.a.s.
2. $\angle DAE = \angle D'A'E'$, $\angle DAF = \angle D'A'F'$. $AB = A'B'$, $AC = A'C'$.	2. Def. $\cong$ $\triangle$s.
3. $\triangle ADE \cong \triangle A'D'E'$. $\triangle ADF \cong \triangle A'D'F'$.	3. a.s.a.
4. $AE = A'E'$, $DE = D'E'$. $AF = A'F'$, $DF = D'F'$.	4. Def. $\cong$ $\triangle$s.
5. Dihed. $\angle VA$ = dihed. $\angle V'A'$.	5. Given.
6. $\angle EDF = \angle E'D'F'$.	6. If dihed. $\angle$s are =, their plane $\angle$s are =. (§ 78)
7. $\triangle DEF \cong \triangle D'E'F'$.	7. s.a.s.
8. $EF = E'F'$.	8. Def. $\cong$ $\triangle$s.
9. $\triangle AEF \cong \triangle A'E'F'$.	9. s.s.s.
10. $\angle EAF = \angle E'A'F'$.	10. Def. $\cong$ $\triangle$s.
11. $\triangle ABC \cong \triangle A'B'C'$.	11. s.a.s.
12. $BC = B'C'$.	12. Def. $\cong$ $\triangle$s.
13. $\triangle BVC \cong \triangle B'V'C'$.	13. s.s.s.
14. $\angle BVC = \angle B'V'C'$.	14. Def. $\cong$ $\triangle$s.
15. Trihed. $\angle V$ = trihed. $\angle V'$.	15. f.f.f. (§ 333)

338. Corollary. On the same sphere or on equal spheres, if two spherical triangles have two sides and the included angle of one equal respectively to the corresponding parts of the other, they are either congruent or symmetric. (s.a.s.)

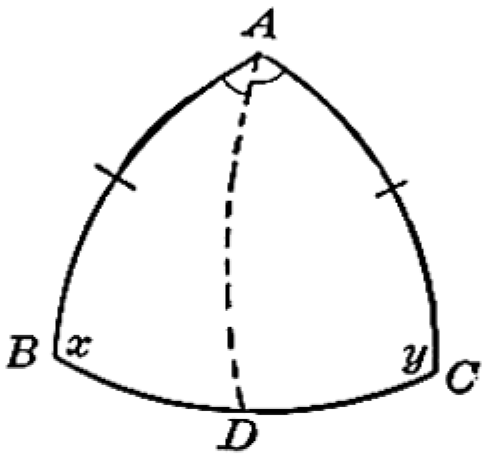
339. THEOREM. On the same sphere or on equal spheres, if two spherical triangles have two angles and the included side of one equal respectively to the corresponding parts of the other, they are either congruent or symmetric. (a.s.a.)

Following the method in § 335, prove that the polar triangles are congruent (or symmetric) by s.a.s.

Draw the figure and write out the proof.

Draw a lune (§ 288) having an angle of about 40°. Now draw a great-circle arc which is perpendicular to one arc of the lune but not to the other. Your lune will then be divided into two spherical triangles that have two angles and a nonincluded side of one equal respectively to the corresponding parts of the other (s.a.a.). Are your triangles congruent or symmetric? How does this compare with plane geometry?

340. THEOREM. If two sides of a spherical triangle are equal, the angles opposite are equal.



GIVEN: In spherical $\triangle ABC$, $AB = AC$.

PROVE: $\angle B = \angle C$.

CONS.: Draw great-circle arc AD bisecting $\angle A$.

PROOF: *Statements*	*Reasons*
1. $AB = AC$.	1. Given.
2. $\angle BAD = \angle DAC$.	2. Cons.
3. $AD = AD$.	3. Identity.
4. $\triangle$s ABD and ACD are symmetric.	4. s.a.s.
5. $\angle B = \angle C$.	5. Def. symmetric spherical $\triangle$s.

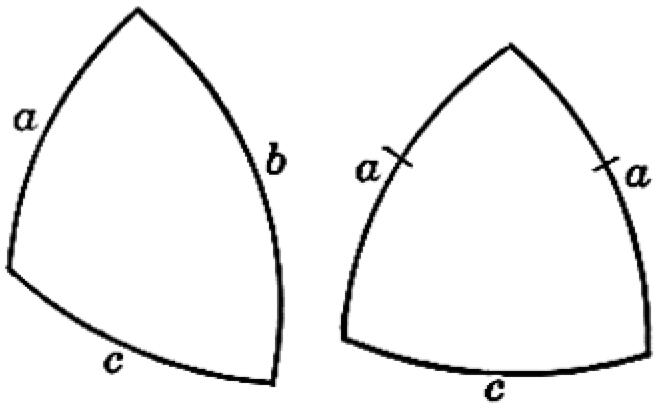
341. CONVERSE THEOREM. If two angles of a spherical triangle are equal, the sides opposite are equal.

First draw the polar triangle and prove that it is isosceles. Draw a figure and write out the proof.

342. Corollary. An equilateral spherical triangle is equiangular; and conversely.

If the sides a, b, and c of the spherical triangle in the figure are read clockwise, the order is a, b, c; counterclockwise, the order is a, c, b. These orders, of course, are different.

If $a = b$, however, both orders are a, a, c. In other words, in an isosceles triangle, the sides may have only one order. The same is true about the angles. We therefore have the following corollary.



343. Corollary. If two symmetric triangles are isosceles, they are congruent.

EXERCISES

1. Prove: A point on a great-circle arc that bisects another great-circle arc at right angles is equidistant from the ends of the second arc.

2. Prove: If a point is equidistant from the ends of a great-circle arc, it lies on the great-circle arc that bisects the first arc at right angles.

3. Prove that the medians to the equal sides of an isosceles spherical triangle are equal.

4. Prove that the bisectors of the base angles of an isosceles spherical triangle are equal.

5. Prove the proposition in spherical geometry which is analogous to the two-point theorem (§ 449) in plane geometry.

6. Construct the bisector of a given spherical angle.

7. Construct the midpoint of a given great-circle arc.

8. Through a given external point construct a great-circle arc perpendicular to a given great-circle arc. Discuss the possible number of solutions.

9. Prove: If the sides of a spherical triangle are proportional to the angles of its polar triangle, both triangles are equilateral.

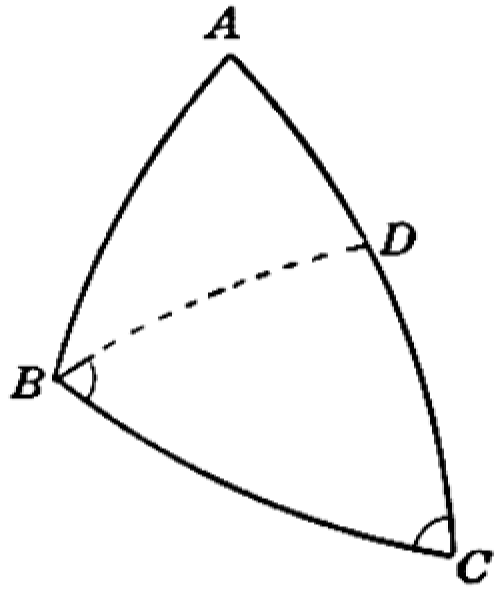
10. Prove: If the diagonals of a spherical quadrilateral bisect each other, the opposite sides are equal.

11. Prove: If the opposite sides of a spherical quadrilateral are equal, the opposite angles are equal.

12. Prove that in a spherical quadrilateral whose opposite sides are equal the diagonals bisect each other.

SUPERIOR WORK

S 1. Prove: If two angles of a spherical triangle are unequal, the sides opposite are unequal in the same order.



Given: In spherical $\triangle ABC$, $\angle B > \angle C$.

Prove: $AC > AB$.

Cons.: Draw great-circle arc BD, making $\angle CBD = \angle C$.
Write out the proof.

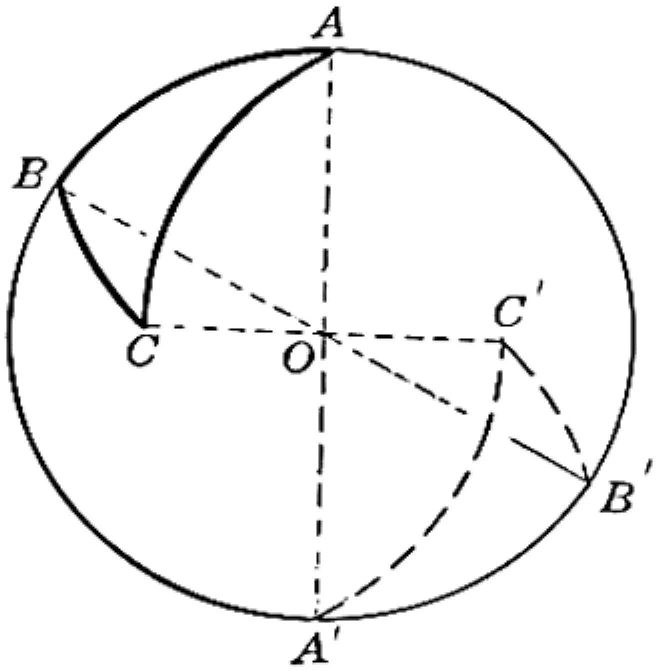
S 2. Using the indirect method, prove the converse of S 1.

AREAS OF SPHERICAL TRIANGLES

In this lesson we shall find the area of a spherical triangle.

344. Definition. *Two spherical triangles are diametrically opposite if their corresponding vertices are the ends of the same diameter.*

345. THEOREM. Two diametrically opposite spherical triangles are symmetric.



Given: Spherical ≜ ABC and $A'B'C'$ on sphere O. AA', BB', and CC' diameters.

Prove: ≜ ABC and $A'B'C'$ symmetric.

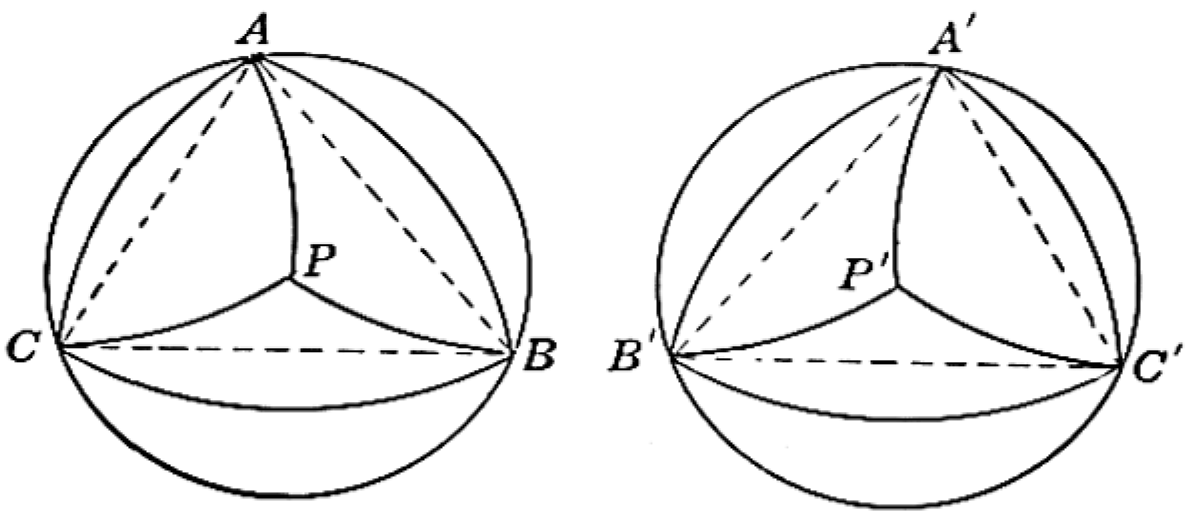
Prove that the sides are respectively equal, and use § 334. Write out the proof.

Since two congruent spherical triangles could be made to enclose the same surface, they have equal areas.

346. Postulate. *Congruent spherical triangles have equal areas.*

In the following theorem we shall prove that two symmetric spherical triangles also have equal areas. We shall do this by dividing each triangle into three isosceles triangles, because in an isosceles triangle the arrangement of the parts does not matter (§ 343).

347. THEOREM. On the same sphere or on equal spheres two symmetric spherical triangles have equal areas.



Given: Symmetric spherical ≜ ABC and $A'B'C'$ on = spheres. (Spheres not drawn.)

Prove: Area $\triangle ABC$ = area $\triangle A'B'C'$.

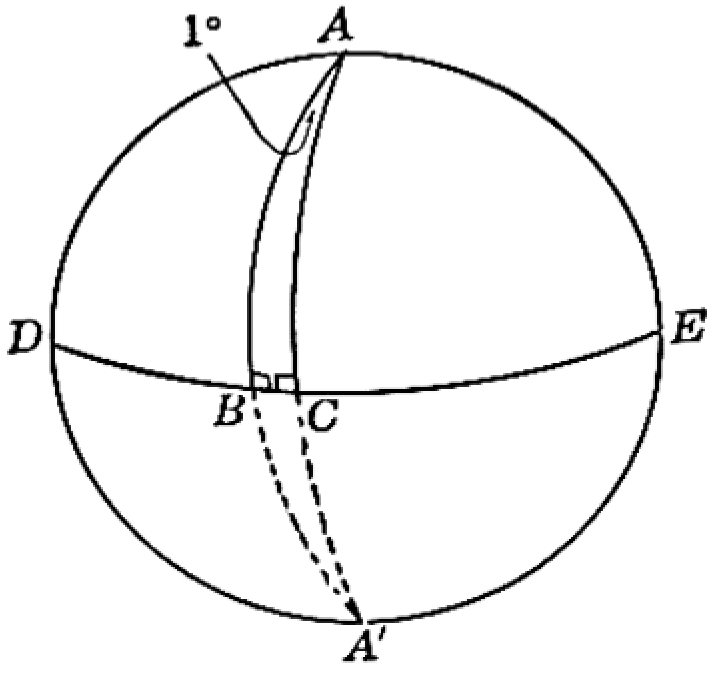
Cons.: Through A, B, and C, draw a small ⊙ whose pole is P. (§ 253)
Through A', B', and C', draw a small ⊙ whose pole is P'.
Draw great-circle arcs PA, PB, PC, $P'A'$, $P'B'$, and $P'C'$.
Draw chords AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

Proof: *Statements*	*Reasons*
1. $\widehat{AB} = \widehat{A'B'}$, $\widehat{BC} = \widehat{B'C'}$, $\widehat{CA} = \widehat{C'A'}$.	1. Def. symmetric ⟁. (§ 331)
2. Chord AB = chord $A'B'$. Chord BC = chord $B'C'$. Chord CA = chord $C'A'$.	2. In = Ⓢ, = arcs have = chords. (§ 511)
3. Plane $\triangle ABC \cong$ plane $\triangle A'B'C'$.	3. s.s.s.
4. $\odot ABC = \odot A'B'C'$.	4. Only one ⊙ can be circumscribed about a $\triangle$. (§ 525)
5. $\widehat{PA} = \widehat{P'A'}$, $\widehat{PB} = \widehat{P'B'}$.	5. On = spheres, = Ⓢ have = polar distances. (§ 306)
6. ⟁PAB and $P'A'B'$ are symmetric.	6. s.s.s. (§ 334)
7. $PA = PB, P'A' = P'B'$.	7. All pts. on a ⊙ of a sphere equidist. from pole. (§ 304)
8. $\triangle PAB \cong \triangle P'A'B'$. Similarly, $\triangle PBC \cong \triangle P'B'C'$, $\triangle PCA \cong \triangle P'C'A'$.	8. If 2 symmetric ⟁ are isosceles they are $\cong$. (§ 343)
9. $\triangle PAB = \triangle P'A'B'$. $\triangle PBC = \triangle P'B'C'$. $\triangle PCA = \triangle P'C'A'$.	9. $\cong$ spherical ⟁ have = areas. (§ 346)
10. Area $\triangle ABC$ = area $\triangle A'B'C'$.	10. Add. Ax.

Note: If the pole P falls outside $\triangle ABC$, we must add two isosceles triangles and subtract the third.

Spherical Degree

In the figure, ABC is a birectangular spherical triangle in which $\angle A = 1°$. Since $\widehat{BC} = 1°$ (§ 313), it can be seen that there are 360 such triangles in the hemisphere ADE, or 720 in the sphere. The area of $\triangle ABC$ is called one *spherical degree.*



348. Definition. *A spherical degree is the area of a birectangular spherical triangle whose third angle is one degree.*

349. Corollary. The area of a sphere equals 720 spherical degrees.

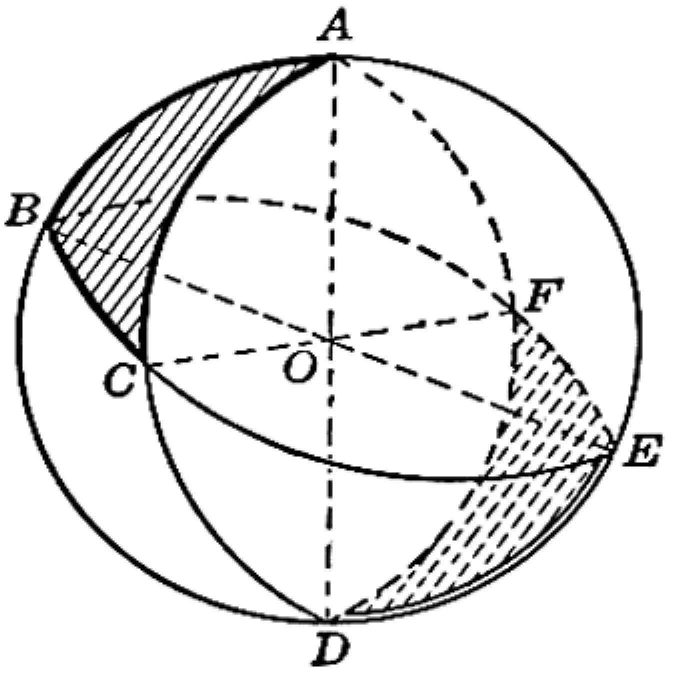
350. If the sum of the angles of a birectangular spherical triangle is 200°, the third angle A equals $200° - 180°$, or 20°. The triangle would therefore contain 20 spherical degrees. Thus it can be seen that the number of spherical degrees in a birectangular spherical triangle is equal to the number of degrees by which the sum of its angles exceeds 180°. We shall prove that this is true for all spherical triangles. The number of degrees by which the sum of the angles of a spherical triangle exceeds 180° is called its *spherical excess.*

351. Definition. *The spherical excess of a spherical triangle is the difference between the sum of the angles of the triangle and 180°.*

If in the figure on page 193, arcs AB and AC are extended to meet at A', the lune $ABA'C$ is formed (§ 288). This lune is twice $\triangle ABC$. In general, a lune contains twice as many spherical degrees as the angular degrees in one angle.

352. Corollary. The number of spherical degrees in a lune equals twice the number of degrees in its angle.

353. THEOREM. The area of a spherical triangle in spherical degrees equals its spherical excess.



GIVEN: Spherical $\triangle ABC$.

A, B, and C represent the number of degrees in the angles A, B, and C.

PROVE: $\triangle ABC = (A + B + C - 180)$ spherical degrees.

CONS.: Extend arcs AB, BC, and CA completing the great circles and meeting again in D, E, and F.

Draw AD, BE, and CF.

PROOF: *Statements*	*Reasons*
1. A and D are in planes ABD and ACD.	1. Cons.
2. O is in planes ABD and ACD.	2. Def. of great ⊙.

3. AOD is a st. line or AD is a diameter. Similarly, BE and CF are diameters.	3. Intersection of 2 planes is a st. line.
4. $\triangle ABC + \triangle BDC =$ lune $ABDC = 2A$ sph. degs. $\triangle ABC + \triangle ACE =$ lune $BAEC = 2B$ sph. degs. $\triangle CDE + \triangle DEF =$ lune $CDFE = 2C$ sph. degs.	4. (§ 352)
5. $2\cdot\triangle ABC + (\triangle BDC + \triangle ACE + \triangle CDE + \triangle DEF)$ $= (2A + 2B + 2C)$ sph. degs.	5. Add. Ax.
6. ⟁ DEF and ABC are symmetric.	6. Diametrically opp. spherical ⟁ are symmetric. (§ 345)
7. $\triangle DEF = \triangle ABC$.	7. On same sphere, symmetric ⟁ are =. (§ 347)
8. $\triangle BDC + \triangle ACE + \triangle CDE + \triangle DEF$ $= \triangle BDC + \triangle ACE + \triangle CDE + \triangle ABC$ $=$ hemisphere $C\text{–}ABDE$ $= 360$ sph. degs.	8. Area sphere = 720 sph. degs. (§ 349)
9. $2\cdot\triangle ABC + 360 =$ $(2A + 2B + 2C)$ sph. degs.	9. Subs. Ax. (In Eq. 5)
10. $\triangle ABC + 180 =$ $(A + B + C)$ sph. degs.	10. Div. Ax.
11. $\triangle ABC = (A + B + C - 180)$ sph. degs.	11. Subt. Ax.

Note that the spherical excess of a spherical triangle is the difference between the sum of the angles of the spherical triangle and the sum of the angles of a plane triangle. The spherical excess of a spherical polygon is likewise the difference between the sum of the angles of the spherical polygon and the sum of the angles of a plane polygon of the same number of sides.

354. Definition. *The spherical excess of a spherical polygon of* n *sides equals the sum of the angles of the polygon less* (n − *2)180.*

355. THEOREM. The area of a spherical polygon equals its spherical excess.

Prove this by drawing all the diagonals from one vertex, dividing the polygon into $(n - 2)$ triangles.

Note: If the area of a spherical surface is known in spherical degrees, its area in square units can be found by using § 349. Thus, if P is the area of a spherical polygon in square units, E is its spherical excess, and S is the area of the sphere in square units, then $\frac{P}{S} = \frac{E}{720}$.

EXERCISES

1. On a sphere whose radius is 10 inches the angle of a lune is 40°. Find the area of the lune in spherical degrees and in square inches.
2. What part of the earth's surface lies between two meridians whose difference in longitude is 15°?
3. Find the area of a spherical triangle whose angles are 70°, 80°, and 120°.
4. Find the area of a birectangular spherical triangle in which the third angle is 70°.
5. What part of a sphere is a trirectangular spherical triangle?
6. Find the number of square miles in a spherical degree on the earth. (Diameter = 8000 miles)
7. Each angle of an equiangular spherical triangle is 120°. What part of the sphere is the triangle?
8. The perimeter of a spherical triangle is 300°. Find the area of its polar triangle.
9. A certain lune equals a spherical triangle whose angles are 90°, 70°, and 70°. Find the angle of the lune.
10. Two angles of a spherical triangle are 60° and 100°. If the triangle equals one-tenth of the sphere, find the third angle.
11. One angle of a spherical quadrilateral is 100° and the other three angles are right angles. Find the area of the quadrilateral.
12. On a sphere of radius 10 inches the area of a spherical triangle is 10π square inches. Find the angle of a lune that has the same area as the triangle.
13. Find the area of a spherical hexagon if each angle is 150°.
14. The area of an equilateral spherical triangle is $\frac{3}{8}$ of the area of the sphere. Find each angle of the triangle.
15. A spherical triangle has $\frac{1}{6}$ the area of a sphere. Find its angles if they are in the ratios of $3:4:5$.
16. Find the number of inches in the perimeter of a trirectangular spherical triangle whose area is 32π square inches.
17. Find the volume of a spherical wedge whose altitude is 12 inches and whose dihedral angle is 24°.
18. The altitude of a spherical pyramid is 8 inches and its dihedral angles are 60°, 70°, and 80°. Find the volume of the spherical pyramid.
19. Two angles of a spherical triangle are 70° and 90°. If the sides of the third angle were extended, they would form a lune having six times the area of the triangle. Find the area of the triangle.
20. On a sphere of 10-inch radius a spherical triangle has an area of 80π square inches. Find the number of inches in the perimeter of the polar triangle.

21. The altitude of a certain zone equals one-third the radius of the sphere. Prove that the area of the zone equals the area of a lune whose angle is 60°.

22. Write a formula for the area of a spherical polygon in terms of its spherical excess E and the radius of the sphere r.

SUPERIOR WORK

S 1. The straight-line distances between the vertices of a spherical triangle are $5\sqrt{2}$, $5\sqrt{2}$, and $5\sqrt{3}$ inches. If the diameter of the sphere is 10 inches, find the area of the spherical triangle.

S 2. Find the area of the surface of the earth north of the 30th parallel between two meridians that make an angle of 150°. *Hint:* This is not a spherical triangle. First find the area of the zone north of the 30th parallel.

S 3. The angles of a spherical triangle are 90°, 90°, and 60°. The angle of a lune on another sphere is 40°. If the area of the spherical triangle equals the area of the lune, find the ratio of the radii of the two spheres.

S 4. In the following table the names of certain plane figures are listed on the left. On the right are the analogous spherical figures. Tell what figures belong in the spaces that have been left blank. If the corresponding figure does not exist, state "None." *Do not write in this book.*

Plane Figure	*Spherical Figure*
Straight line	Great circle
Line segment	—?—
Parallel lines	None
Circle	Small circle
Circular arc	—?—
Center of circle	Pole of small circle
—?—	Pole of great circle
—?—	Polar distance of small circle
Angle	—?—
Polygon	—?—
—?—	Lune
Right triangle	—?—
—?—	Birectangular triangle
—?—	Trirectangular triangle
Similar triangles	—?—

S 5. State the proposition in spherical geometry that corresponds to each of the following propositions in plane geometry:

a. A straight line is the shortest distance between two points.

b. Two straight lines can intersect in only one point.

c. The sum of the angles of a triangle equals 180°.

d. An exterior angle of a triangle equals the sum of the two remote interior angles.
e. Radii of the same circle are equal.
f. Through a given point, only one perpendicular can be drawn to a given line.
g. The area of a triangle equals half the base times the altitude.
h. If the angles of one triangle are equal respectively to the angles of another triangle, the triangles are similar.
i. The area of a circle equals πr^2. (Page 161, S 3.)
j. If one transversal to two lines makes the corresponding angles equal, any other transversal will do likewise.

SUMMARY

356. On a sphere great circles and small circles are analogous to straight lines and circles respectively in a plane. Distances are measured along great circles. The pole of a small circle corresponds to the center of a circle in a plane; the polar distance of the small circle corresponds to the radius of a circle in a plane. As in the case of a central angle in plane geometry, if the vertex of a spherical angle is the pole of a circle, the spherical angle is measured by the intercepted arc.

The breakdown in the analogy between spherical and plane figures comes in the relationships among the angles. The sum of the angles of a spherical triangle is not constant but varies between 180° and 540°. In fact, the excess over 180° measures the area of the spherical triangle in spherical degrees. Since the angles of a spherical triangle are independent magnitudes, they determine a spherical triangle. Spherical triangles are congruent (or symmetric) if they agree in *a.a.a.*, *s.s.s.*, *s.a.s.*, or *a.s.a.* The sides and the angles of a spherical polygon are measured by the face angles and dihedral angles respectively of a corresponding polyhedral angle. Trihedral angles are therefore congruent or symmetric under the same conditions as spherical triangles.

The analogy between a plane tangent to a sphere and a line tangent to a circle holds in the relation between the plane and the diameter to the point of contact. As in the case of a circle and a triangle, a sphere may be inscribed in or circumscribed about any tetrahedron.

The area of a sphere equals the area of four of its great circles. The area of a zone is the same part of the area of the sphere as the altitude of the zone is of the diameter of the sphere. The area of a spherical polygon is the same part of the area of the sphere as the spherical excess of the polygon is of 720.

$$\text{Sphere } S = 4\pi r^2$$

$$\frac{\text{Zone } Z}{\text{Sphere } S} = \frac{h}{d} \quad \begin{array}{l}\text{altitude of zone}\\ \text{diameter of sphere}\end{array}$$

$$\frac{\text{Spherical polygon } P}{\text{Sphere } S} = \frac{E}{720} \quad \text{spherical excess}$$

The space enclosed by a sphere and a surface generated by rotating a ray about the center of the sphere is a spherical pyramid, a spherical cone, a spherical sector, or a spherical wedge. The volume of any of these solids, as well as the volume of the whole sphere, equals one-third the product of the spherical surface and the radius of the sphere. This is analogous to the volume of a pyramid or a cone.

RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, see section indicated.

- sphere § 245
  - center, radius
  - chord, diameter
- circles of sphere § 249
  - great
  - small
  - axis, poles
- hemisphere § 256
- plane tangent to sphere § 259
- line tangent to sphere § 259
- tangent spheres § 261
- sphere inscribed in polyhedron § 266
- sphere circumscribed about polyhedron § 267
- sphere inscribed in cylinder or cone § 268
- sphere circumscribed about cylinder or cone § 269
- zone § 277
  - bases, altitude § 279
  - of one base § 279
  - of two bases § 279
- spherical pyramid or cone § 284
  - base, altitude
- spherical sector § 285
  - base, altitude
- spherical wedge § 287
  - base, altitude
- lune § 288
- spherical segment § 295
  - bases, altitude
  - of two bases
  - of one base
- spherical distance § 303
- polar distance § 305
- quadrant § 307
- spherical angle § 310
  - sides, vertex § 310
  - measure § 311
  - adjacent, vertical § 315
  - complementary, supplementary § 315
  - obtuse, acute § 315
- spherical polygon § 316
- spherical triangle § 319
  - isosceles, equilateral § 319
  - equiangular § 319
  - median, altitude § 320
  - right, oblique § 329
  - birectangular, trirectangular § 329
- polar triangle § 323
- spherical triangles
  - congruent, symmetric § 331
  - diametrically opposite § 344
- spherical degree § 348
- spherical excess
  - of triangle § 351
  - of polygon § 354

REVIEW EXERCISES

1. On a sphere of 12-inch radius find the spherical distance between the ends of a 12-inch chord.

2. A zone has one-fourth the area of a sphere. If the altitude of the zone is 3 inches, find the diameter of the sphere.

3. The area of a great circle of a sphere is 80 square inches. Find the area of a trirectangular triangle on that sphere.

4. The ratio of the diameters of two hemispherical solids is 3:5. Find the ratio of their total areas.

5. The area of a 36° lune equals 10π square inches. Find the radius of the sphere.

6. One side of a spherical triangle equals 170°. Between what limits is the sum of the other two sides?

7. One angle of a spherical triangle is 20°. Between what limits is the sum of the other two angles?

8. One angle of a spherical pentagon is 108°. If the other four angles are equal, between what limits is each?

9. Prove that if a great circle passes through a pole of a small circle, the planes of the two circles intersect at right angles.

10. Prove: If two chords of a sphere are equal, they subtend equal spherical distances.

11. Prove: If two parallel lines intersect a sphere, they intercept equal spherical distances.

12. Construct a sphere that will contain a given circle and a point in another plane.

13. The volume of a spherical wedge is 6π cubic inches and its edge is 6 inches. Find its dihedral angle.

14. The total area of a tetrahedron is 120 square inches and its volume is 200 cubic inches. Find the radius of the inscribed sphere.

15. **a.** Prove that the sum of the n dihedral angles of a polyhedral angle is greater than $(n - 2)$ straight angles but less than n straight angles.
    b. If the face angles of a trihedral angle are equal, between what limits is each dihedral angle?

16. Prove: If two sides of a spherical triangle are quadrants, the median to the third side divides the figure into two congruent isosceles triangles.

17. The angles of a spherical triangle are 80°, 80°, and 40°. Find the area of the triangles formed by the bisector of the 40° angle.

18. Prove that if one angle of an equilateral spherical triangle equals 150°, the area of the triangle equals the area of three trirectangular triangles.

19. The plane of a small circle of a sphere is 24 inches from the center. If the area of the circle is 49π, find the volume of the sphere.

20. A lune whose angle is 30° is cut by the arc of a great circle making the sum of two interior angles on the same side of the arc equal to 170°. Find the ratio of the areas of the two triangles.

21. The angles of a spherical triangle are 80°, 90°, and 50°. Another triangle having twice the area has its angles in the ratios 3:4:6. Find these angles.

22. The radius of a sphere is 10 inches. Find the area of a circle 6 inches from the center of the sphere and the area of the smaller spherical surface bounded by the circle.

23. On a sphere of 12-inch radius the area of a spherical triangle equals 48π square inches. Find the number of inches in the perimeter of its polar triangle.

24. The angles of a spherical triangle are 90°, 90°, and 70°. Find the area of its polar triangle.

25. The angles of an equiangular spherical polygon are each 156°. If its area is one-third of the sphere, how many sides does the polygon have?

26. Find the number of square miles in the earth's surface between the 30° and 45° circles of latitude. ($d = 8000$ miles.)

27. Prove: If two sides of a spherical triangle are quadrants, the trisector of the angle between them also trisects the third side.

28. The radius of a sphere is 5 feet. From a point 13 feet from the center, lines are drawn tangent to the sphere. Find the length of the locus of the points of contact.

29. Two spherical triangles on different spheres have the angles of one equal respectively to the angles of the other. The area of one triangle is 30 square inches, but the area of the second is 270 square inches. Find the ratio of the radii of the spheres.

30. A right circular cone is inscribed in a sphere. The altitude of the cone is 8 inches and the radius of its base is 4 inches. Find the volume of the sphere outside the cone.

31. Two planes perpendicular to a diameter of a sphere trisect the area of the sphere. If the whole diameter is 6 inches, find its segments.

32. A spherical triangle whose angles are 100°, 90°, and 70° equals a zone. Find the ratio of the altitude of the zone to the diameter of the sphere.

33. The volume of a spherical pyramid is 3π cubic inches, and its dihedral angles are 80°, 60°, and 100°. Find its altitude.

34. The area of a trirectangular triangle on a sphere is 40 square inches. Find the area of a 45° lune on another sphere whose diameter is three times as large.

35. Prove: If two great circles are perpendicular to a third great circle, they cannot both be perpendicular to a fourth great circle.

36. On a certain sphere the angles of one spherical triangle are respectively twice the angles of another spherical triangle. If one triangle has five times the area of the other, find the area of each.

37. A solid is made by cutting two wooden spheres in equal circles and gluing the flat surfaces of the smaller parts together. The radii of the spheres are 20 inches and 15 inches, and the radius of the circular cut is 12 inches. Find the area of the surface of the finished solid.

SUPERIOR WORK

S 1. Prove that the sum of the exterior angles of a spherical polygon is less than 360°.

S 2. The altitude of a zone is 4 inches and the radii of its bases are 7 inches and 15 inches. Find the area of the zone.

S 3. Find the number of square miles on the earth bounded by the equator, the 30th parallel and by two meridians that form an angle of 12°. (See page 197, S 2.)

S 4. Find the area of a portion of the earth's surface bounded by parallels 30° S and 45° N and the meridians 75° W and 100° W.

S 5. The straight-line distances from the poles to a circle of a sphere are 2 inches and 3 inches. Find the area of the circle and the areas of the two zones into which the sphere is divided. (See page 161, S 3.)

S 6. The slant height of a right circular cone is s, and the radius of its base is r. Prove that the radius of the inscribed sphere equals $r\sqrt{\frac{s-r}{s+r}}$.

S 7. Find the volume of the solid described in Exercise 37 above.

S 8. Two planes perpendicular to a diameter of a sphere trisect the volume of the sphere. If the whole diameter of the sphere is 6 inches, find its segments.

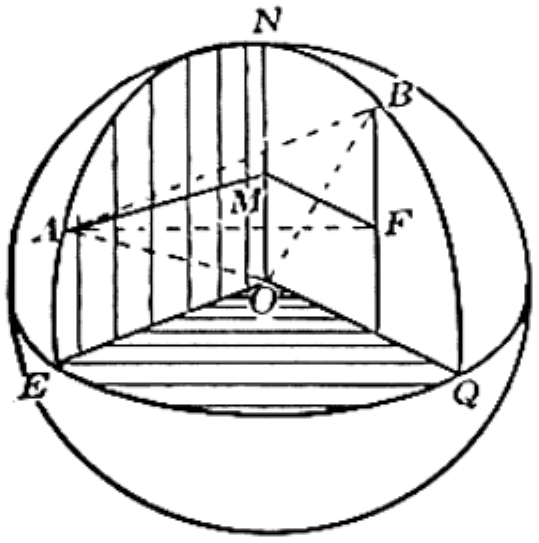
Suggestion: Using § 297, solve the resulting equation by graphing.

S 9. *Distance along a Great Circle.* In modern long-distance travel the following problem is of increasing importance.

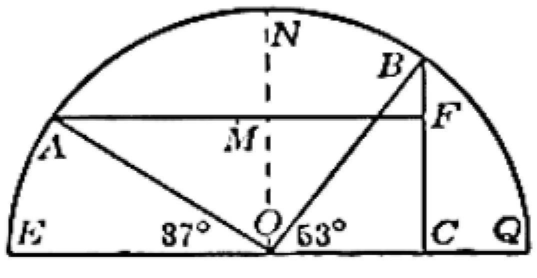
Problem. Find the great circle distance that a plane flies from San Francisco (Latitude 37° N, Longitude 122° W) to Liverpool (Latitude 53° N, Longitude 3° W).

a. Figure (a) is a space diagram. A is San Francisco, and B is Liverpool. Imagine that a section of the earth has been cut away. The cuts were made along the meridians NE and NQ and along the equator EQ. O is the center of the earth, and ON is half the earth's axis. AM is drawn parallel to EO, and MF is drawn parallel to OQ.

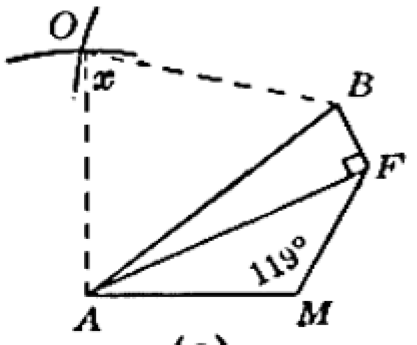
$\angle EOA$ = latitude of $A = 37°$; $\angle QOB$ = latitude of $B = 53°$. $\angle AMF = \angle EOQ$ = difference in longitude between A and $B = 122° - 3° = 119°$. ABF is a right triangle.



(a)



(b)



(c)

b. Copy Figure (b) on a larger scale. The radius of the semicircle does not matter, but you must construct $\angle EOA = 37°$ and $\angle QOB = 53°$ with a protractor.

Cut out this semicircle and fold it along ON to form an angle of about 119°. This will give you a space picture of the problem. You will observe AMF will not then be a straight line.

To complete the solution, you must construct the line segment AF in space, then AB, and finally the central angle AOB (Figure a).

c. Figure (c): Construct the $\triangle AMF$ by copying AM and MF from Figure (b) and making $\angle AMF = 119°$.

Construct BF perpendicular to AF, copying BF from Figure (b), and complete $\triangle ABF$.

Finally, construct an isosceles triangle on AB as the base, making the legs AO and OB as in Figure (b).

Measure $\angle O$. The number of *minutes* in $\angle O$ is your answer in nautical miles. (Page 178, Exercise 13.)

Note: You can compute $\angle O$ by using a table of cosines and the Law of Cosines. Explain how you would figure out the following quantities, after letting $OA = 1$.

In Figure (b) AM, MF, BC, MO, and BF.

In Figure (c) AF, AB, and $\angle O$.

However, the method would be rather long. By trigonometry and a special table of "haversines" the work can be greatly shortened.

S 10. Find the distance from New York (Lat. 41° N, Long. 74° W) to Paris (Lat. 49° N, Long. 2° E). This was Lindbergh's flight in 1927.

TEST

1. Assuming that the diameter of the earth is 8000 miles, find the length of the 30th parallel.

2. Prove that a parallelogram inscribed in a sphere is a rectangle.

3. A right circular cylinder has its altitude 12 feet and the radius of its base 6 feet. Find the area and the volume of the inscribed sphere.

4. One angle of a spherical triangle equals 90° and the other two angles are in the ratio of 1:2. If the area of the triangle equals $\frac{1}{12}$ of the area of the sphere, find the angles of the triangle.

5. The angles of a spherical triangle are 100°, 80°, and 60°. The radius of the sphere is 18 inches. Find the altitude of a zone having the same area as the triangle.

6. On a sphere of radius 9 inches the perimeter of a spherical triangle equals 6π inches. Find the number of square inches in the area of its polar triangle.

7. A 45° sector is revolved about one of its radii which are 10 inches each. Find the total surface of the solid generated.

8. The radius of a sphere is 5 inches. Find the volume of the spherical segment cut off by a plane 3 inches from the center.

9. Which of the following statements are true and which are false?
   - a. Three points on a sphere do not always determine a small circle.
   - b. If two chords of a sphere are perpendicular to a third chord at its ends, they are equal.
   - c. The opposite angles of a quadrilateral inscribed in a sphere are supplementary.
   - d. If two points are diametrically opposed, they are "poles apart."
   - e. If a line is tangent to a sphere, it is tangent to all the circles of the sphere passing through the point of contact.
   - f. As a semicircle revolves about its diameter two of its equal arcs generate equal zones.
   - g. The sides of a spherical quadrilateral cannot be quadrants.
   - h. A spherical triangle cannot have the angles 120°, 100°, and 30°.
   - i. At Chicago the length of a degree of longitude is not equal to the length of a degree of latitude.
   - j. Two angles of a spherical triangle determine the third angle.

10. Complete each of the following statements.
    - a. If the diameter of a circle is also the diameter of the sphere, the circle is a(n) —?—.
    - b. If two spheres are tangent to the same —?—, their radii to the points of contact are parallel.
    - c. If two angles of a spherical triangle are 50° and 120°, the third angle must be less than —?— degrees.
    - d. If two zones on the same sphere have equal altitudes, they are —?—.
    - e. If a zone has two bases, one of them is a(n) —?—.
    - f. A figure formed by two meridians is called a(n) —?—.
    - g. Each angle of an equiangular spherical triangle is between —?— degrees and —?— degrees.
    - h. A lune is so named because it resembles the varying shapes of the —?—.
    - i. If the radius of a sphere increases, a spherical degree on it —?—.
    - j. A trirectangular triangle has —?— the area of a hemisphere.

Useful Formulas

NOTATION

A	angle of lune
B, B'	area of base
C, C'	circumference
d	diameter
E	spherical excess of polygon $= \text{sum of } \measuredangle - (n - 2)180^\circ$
h, h'	altitude
l	length
L, L'	lateral area
M	area of midsection of prismatoid
p, p'	perimeter of base
r, r'	radius
r_1, r_2	radius of base of zone
s, s'	slant height
S	area
T, T'	total area
V, V'	volume
w	width
Z, Z'	area of zone

FORMULAS

Cone	$V = \frac{1}{3}Bh$	(§ 234)
Cone of revolution	$L = \frac{1}{2}sC$	(§ 219)
	$L = \pi rs$	(§ 219)
	$T = \pi r(r + s)$	(§ 220)
Cones of revolution, similar	$\frac{L}{L'} = \frac{T}{T'} = \frac{h^2}{h'^2} = \frac{s^2}{s'^2} = \frac{r^2}{r'^2}$	(§ 236)
	$\frac{V}{V'} = \frac{h^3}{h'^3} = \frac{s^3}{s'^3} = \frac{r^3}{r'^3}$	(§ 236)
Cylinder	$V = Bh$	(§ 181)
Cylinder of revolution	$L = hC$	(§ 156)
	$L = 2\pi rh$	(§ 156)
	$T = 2\pi r(r + h)$	(§ 157)
Cylinders of revolution, similar	$\frac{L}{L'} = \frac{T}{T'} = \frac{h^2}{h'^2} = \frac{r^2}{r'^2}$	(§ 184)
	$\frac{V}{V'} = \frac{h^3}{h'^3} = \frac{r^3}{r'^3}$	(§ 184)
Frustum of cone	$V = \frac{1}{3}h(B + B' + \sqrt{BB'})$	(§ 240)
	$V = \frac{1}{3}\pi h(r^2 + r'^2 + rr')$	(Page 134, Exercise 29)
Frustum of cone of revolution	$L = \frac{1}{2}s(C + C')$	(§ 228)
	$L = \pi s(r + r')$	(§ 228)
Frustum of pyramid	$V = \frac{1}{3}h(B + B' + \sqrt{BB'})$	(§ 240)
Frustum of regular pyramid	$L = \frac{1}{2}s(p + p')$	(§ 227)
Lune	$S = \frac{\pi A r^2}{90}$	(§ 352)
Prism	$V = Bh$	(§ 180)
Prism, right	$L = hp$	(§ 156)
Prismatoid	$V = \frac{1}{6}h(B + 4M + B')$	(§ 242)

Pyramid	$V = \frac{1}{3}Bh$	(§ 234)
Pyramid, regular	$L = \frac{1}{2}sp$	(§ 218)
Rectangular solid	$V = lwh$	(§ 172)
	Diagonal $= \sqrt{l^2 + w^2 + h^2}$	(§ 165)
Sphere	$T = \pi d^2 = 4\pi r^2$	(§ 274)
	$V = \frac{1}{6}\pi d^3 = \frac{4}{3}\pi r^3$	(§ 291, § 293)
Spherical cone	$V = \frac{1}{3}Br$	(§ 290)
Spherical polygon	$S = \frac{\pi E r^2}{180}$	(§ 355)
Spherical pyramid	$V = \frac{1}{3}Br$	(§ 290)
Spherical sector	$V = \frac{1}{3}Br$	(§ 290)
Spherical segment of one base	$V = \frac{\pi}{3} h^2(3r - h)$	(§ 297)
	$V = \frac{1}{6}\pi h(3r_1^2 + h^2)$	(Page 171, S 6)
Spherical segment of two bases	$V = \frac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2)$	(Page 171, S 6)
Spherical wedge	$V = \frac{1}{3}Br$	(§ 290)
Zone	$Z = 2\pi rh$	(§ 281)
Zones on same sphere	$\frac{Z}{Z'} = \frac{h}{h'}$	(§ 282)

References to Plane Geometry

DEFINITIONS

357. A proposition that is accepted without proof is called an axiom or a postulate.

358. A proposition that is accepted only after it has been proved is called a theorem.

359. A corollary is a theorem that is easily proved from another proposition.

360. The converse of a proposition is a new proposition formed by interchanging the hypothesis and the conclusion.

361. A reversible proposition is a proposition whose converse is equally valid.

362. If the hypothesis and the conclusion of one proposition are the opposites of the hypothesis and the conclusion of another proposition, then each proposition is said to be the opposite of the other proposition.

363. Every definition must be expressed in terms that have been accepted and must be reversible. A good definition should be clear, concise, and not redundant.

364. Induction is the reasoning from the particular to the general; deduction, from the general to the particular.

365. The locus of points is a figure that contains all the points that satisfy a certain condition and no others. The (dynamic) locus of a point is the path of a point that moves while obeying some law.

366. A statement that a is greater than b or a is less than b is called an inequality.

367. A line segment is the part of a straight line between two points.

368. A ray is the part of a straight line to one side of a point.

369. An angle is a figure that is formed by two rays from a common point.

370. A straight angle is an angle whose sides form a straight line.

371. A right angle is one-half as large as a straight angle.

372. An acute angle is less than a right angle.

373. An obtuse angle is an angle greater than a right angle but less than a straight angle.

374. Complementary angles are two angles whose sum equals a right angle.

375. Supplementary angles are two angles whose sum equals a straight angle.

376. Adjacent angles are two angles that have a common vertex and a common side between them.

377. Vertical angles are two nonadjacent angles formed by two intersecting straight lines.

378. Perpendicular lines are two straight lines that meet at right angles.

379. Parallel lines are two straight lines in the same plane that do not meet no matter how far they are extended.

380. Oblique lines are straight lines that are neither perpendicular nor parallel.

381. The distance from a point to a line is the length of the perpendicular from the point to the line.

382. A broken line is a figure in which a number of points not in a straight line are connected in succession by line segments.

383. A polygon is a closed broken line.

384. A diagonal of a polygon is a line segment connecting two nonconsecutive vertices of the polygon.

385. An exterior angle of a polygon is an angle formed by one side of the polygon and the prolongation of an adjacent side.

386. A regular polygon is a polygon that is equiangular and equilateral.

387. The center of a regular polygon is the center of the circle circumscribed about the polygon. The radius of a regular polygon is a line segment from the center to a vertex of the polygon. An apothem of a regular polygon is a perpendicular from the center of the polygon to one of the sides.

388. Congruent polygons are polygons that have all the parts of one equal respectively to the corresponding parts of the other.

389. Two polygons are similar if the angles of one are equal respectively to the angles of the other, and the sides of one are proportional to the corresponding sides of the other.

390. A quadrilateral is a polygon with four sides.

391. A trapezoid is a quadrilateral that has two, and only two, sides parallel.

392. An isosceles trapezoid is a trapezoid with equal legs.

393. A parallelogram is a quadrilateral whose opposite sides are parallel.

394. An altitude of a trapezoid or a parallelogram is a line segment included between two parallel sides and perpendicular to one of them.

395. A rectangle is a parallelogram with a right angle.

396. A rhombus is a parallelogram with two adjacent sides equal.

397. A square is a rectangle with two adjacent sides equal.

398. A triangle is a polygon having three sides.

399. An isosceles triangle is a triangle that has two equal sides.

400. An equilateral triangle is a triangle with three equal sides.

401. A scalene triangle is a triangle with no two sides equal.

402. A right triangle is a triangle that has a right angle.

403. An oblique triangle is a triangle that does not have a right angle.

404. An obtuse triangle is a triangle that has an obtuse angle.

405. An acute triangle is a triangle in which all the angles are acute.

406. An equiangular triangle is a triangle in which all the angles are equal.

407. A median of a triangle is a line segment from a vertex to the mid-point of the opposite side.

408. An altitude of a triangle is a line segment from a vertex perpendicular to the opposite side.

409. A circle is a closed plane curve of which every point is the same distance, called the radius, from a point within, called the center.

410. A chord of a circle is a line segment that connects any two points on the circle.

411. A diameter of a circle is a chord that passes through the center of the circle.

412. A minor arc is an arc less than a semicircle; a major arc is greater than a semicircle.

413. Equal circles are circles that have equal radii.

414. Concentric circles are circles that have the same center.

415. A tangent to a circle is a straight line that touches the circle in only one point.

416. A secant of a circle is a straight line that cuts the circle in two points.

417. A central angle is an angle formed by two radii of a circle.

418. An inscribed angle is an angle formed by two chords meeting on a circle.

419. A polygon is inscribed in a circle if all its vertices are on the circle: the circle is then circumscribed about the polygon.

420. A circle is inscribed in a polygon if it is tangent to all the sides of the polygon; the polygon is then circumscribed about the circle.

421. A common external tangent to two circles is a line that is tangent to two circles on the same side of the line. A common internal tangent to two circles is a line that is tangent to two circles on the opposite sides of the line.

422. Two circles are tangent to each other if they are tangent to the same line at the same point.

423. A sector of a circle is a figure formed by two radii of the circle and the arc they intercept.

424. A segment of a circle is a figure formed by a chord of the circle and its arc.

425. A unit of area is a square whose side is the unit of length.

426. The area of a surface is the number of unit squares that the surface contains.

427. A proportion is a statement of the equality of two ratios (fractions).

428. If the means of a proportion are equal, each mean is said to be the mean proportional between the extremes.

429. If the ratio of two variables is constant, one variable is said to vary as the other.

430. If from a point on one side of a given angle a perpendicular to the other side is drawn, then the leg of the resulting right triangle opposite the given angle divided by the hypotenuse is the ratio called the sine of the given angle; the ratio of the adjacent leg to the hypotenuse is the cosine; the ratio of the opposite leg to the adjacent leg is called the tangent.

AXIOMS AND POSTULATES

431. *Addition Axiom.* If equals are added to equals, the sums are equal.

432. *Subtraction Axiom.* If equals are subtracted from equals, the remainders are equal.

433. *Multiplication Axiom.* If equals are multiplied by equals, the products are equal.

434. *Division Axiom.* If equals are divided by equals (except 0), the quotients are equal.

435. *Substitution Axiom.* A quantity may be substituted for its equal in an expression without changing the value of the expression.

436. A proposition and its opposite-converse are equally valid.

437. *Two-Point Postulate.* Two points determine a straight line.

438. All straight angles are equal.

439. Any proposition about a regular polygon that does not depend on the number of sides is equally valid for the circumscribed circle.

440. The whole of any magnitude is greater than any of its parts.

441. A straight line is the shortest distance between two points.

442. If unequals are added to equals or to unequals, the sums are unequal in the same order.

443. If equals are subtracted from unequals, the remainders are unequal in the same order.

444. If unequals are subtracted from equals, the remainders are unequal in the opposite order.

OTHER PROPOSITIONS

Angles and Lines

445. All right angles are equal.

446. The sum of all the angles about a point in a plane equals 360°.

447. Supplements or complements of the same or equal angles are equal.

448. Vertical angles are equal.

449. *Two-Point Theorem.* Two points each equidistant from the ends of a line segment determine the perpendicular bisector of the line segment.

450. The perpendicular bisector of a line segment is the locus of a point equidistant from the ends of the line segment.

451. The bisector of an angle is the locus of a point equidistant from the sides of the angle.

452. The locus of a point equidistant from two intersecting lines is made up of the bisectors of the angles formed by these lines.

453. The locus of a point at a given distance from a given line is made up of two lines parallel to the given line at the given distance from it.

454. The perpendicular is the shortest distance from a given point to a given line.

Parallel Lines

455. If two lines are cut by a transversal and the corresponding angles are equal, the lines are parallel.

456. If two lines are cut by a transversal and the alternate interior angles are equal, the lines are parallel.

457. If two lines are cut by a transversal and the interior angles on the same side of the transversal are supplementary, the lines are parallel.

458. If two parallel lines are cut by a transversal, then the

(1) corresponding angles are equal.
(2) alternate interior angles are equal.
(3) interior angles on the same side are supplementary.

459. If two lines are perpendicular to the same line, they are parallel to each other.

460. If two lines are parallel to the same line, they are parallel to each other.

461. The locus of a point equidistant from two parallel lines is another line midway between them.

462. If a line is perpendicular to one of two parallel lines, it is perpendicular to the other.

463. Three or more parallels cut proportional segments on two transversals.

464. If three or more parallels cut equal parts on one transversal, they cut equal parts on any other transversal.

465. If a line is parallel to one side of a triangle, it divides the other two sides proportionally.

466. If a line divides two sides of a triangle proportionally, it is parallel to the third side.

Triangles

467. The sum of the angles of a triangle is 180°.

468. An exterior angle of a triangle equals the sum of the two remote interior angles.

469. If two angles of one triangle are equal respectively to two angles of another triangle, the remaining angles are equal.

470. Two triangles are congruent if they agree in

(1) one side and two corresponding angles. (a.s.a., s.a.a.)
(2) two sides and the included angle. (s.a.s.)
(3) two sides and the right angle opposite one of them. (h.l.)
(4) three sides. (s.s.s.)

471. If two sides of one triangle are equal to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second.

472. If two sides of one triangle are equal to two sides of another triangle, but the third side of the first is greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

473. If two sides of a triangle are equal, the angles opposite are equal; and conversely.

474. If two sides of a triangle are unequal, the angles opposite are unequal in the same order; and conversely.

475. The area of a triangle equals half the base times the altitude.

476. If the sides of a triangle are a, b, and c, its area equals $\sqrt{s(s-a)(s-b)(s-c)}$, where s equals half $(a+b+c)$. (Heron's Formula)

477. If an altitude of one triangle equals an altitude of another triangle, the triangles are proportional to the bases.

478. If an angle of one triangle equals an angle of another triangle, the triangles are proportional to the products of the sides forming the equal angles.

479. Two triangles are similar if they
(1) agree in two angles.
(2) agree in one angle and have the including sides proportional.
(3) have all their sides proportional.
(4) have their sides respectively parallel or perpendicular.

480. The areas of two similar triangles are proportional to the squares of the corresponding sides.

Secondary Parts of a Triangle

481. The altitudes of a triangle meet in a point.

482. The medians of a triangle meet in a point that is two-thirds from each vertex to the middle of the opposite side.

483. The angle bisectors of a triangle meet in a point that is equidistant from the sides of the triangle.

484. The perpendicular bisectors of the sides of a triangle meet in a point that is equidistant from the vertices of the triangle.

485. The line segment between the midpoints of two sides of a triangle is parallel to the third side and equal to half of it.

Right Triangles

486. The median to the hypotenuse of a right triangle equals half the hypotenuse.

487. The square of the hypotenuse of a right triangle equals the sum of the squares of the legs (Pythagorean Theorem).

488. If the square of one side of a triangle equals the sum of the squares of the other two sides, the figure is a right triangle (converse of Pythagorean Theorem).

489. If the altitude is drawn to the hypotenuse of a right triangle, it is the mean proportional between the segments of the hypotenuse; and either leg is the mean proportional between the whole hypotenuse and the adjoining segment.

490. If one angle of a right triangle is 60°, the hypotenuse is twice the leg adjacent to the 60° angle.

491. If the hypotenuse of a right triangle is twice one leg, the included angle is 60°.

Polygons

492. The sum of the interior angles of a polygon of n sides is $(n - 2)$ straight angles.

493. The sum of the exterior angles of a polygon is two straight angles.

494. The area of a polygon circumscribed about a circle equals half the product of the perimeter of the polygon and the radius of the circle.

495. The perimeters of two similar polygons are proportional to the corresponding sides; but their areas are proportional to the squares of the corresponding sides.

Quadrilaterals

496. A quadrilateral is a parallelogram if

(1) the opposite sides are equal.
(2) two sides are equal and parallel.
(3) the opposite angles are equal.
(4) the diagonals bisect each other.

497. In a parallelogram

(1) the opposite sides are equal.
(2) the opposite angles are equal.
(3) the diagonals bisect each other.

498. The area of a trapezoid equals the average of its bases times the altitude.

499. The line segment connecting the midpoints of the legs of a trapezoid is parallel to the bases and equal to half their sum.

500. The base angles of an isosceles trapezoid are equal.

501. If the base angles of a trapezoid are equal, the trapezoid is isosceles.

502. The diagonals of an isosceles trapezoid are equal.

503. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

504. The diagonals of a rectangle are equal.

505. If the diagonals of a parallelogram are equal, the figure is a rectangle.

506. A diagonal of a parallelogram divides it into two congruent triangles.

507. The diagonals of a rhombus are perpendicular.

508. The area of a rhombus equals half the product of its diagonals.

509. The area of a parallelogram equals the product of its base and altitude.

Circles

510. In the same circle or in equal circles equal central angles have equal arcs; and conversely.

511. In the same circle or in equal circles equal chords have equal arcs; and conversely.

512. In the same circle or in equal circles if two chords are unequal, their minor arcs are unequal in same order; and conversely.

513. In the same circle or in equal circles equal chords are equally distant from the center; and conversely.

514. A diameter bisects a circle.

515. A line perpendicular to a diameter at one end is tangent to the circle.

516. A diameter is perpendicular to a tangent at the point of contact.

517. A central angle equals the intercepted arc in degrees.

518. If the sides of an angle cut or touch a circle, the angle equals half the algebraic sum of the arc degrees between the sides and the extensions of the sides of the angle — one arc being negative if the vertex is outside the circle.

519. An inscribed angle, or an angle between a tangent and a chord to the point of contact, equals half the intercepted arc in degrees.

520. Angles inscribed in the same arc are equal.

521. An angle inscribed in a semicircle is a right angle.

522. Tangents to a circle from an external point are equal and make equal angles with the line joining the point to the center.

523. The common external or internal tangents to two circles are equal.

524. If two circles are tangent, the line of centers passes through the point of contact.

525. A circle, and only one circle, can be circumscribed about or inscribed in any triangle or any regular polygon.

526. In an inscribed quadrilateral the opposite angles are supplementary.

527. In a circumscribed quadrilateral the sum of two opposite sides equals the sum of the other two sides.

528. If the opposite angles of a quadrilateral are supplementary, a circle can be circumscribed about it.

529. If the sum of two opposite sides of a quadrilateral equals the sum of the other two sides, a circle can be inscribed in it.

530. If two chords intersect within a circle, the product of the segments of one chord equals the product of the segments of the other chord.

531. If a chord is perpendicular to a diameter, either segment of the chord is the mean proportional between the segments of the diameter.

532. If two secants are drawn to a circle, the product of one secant and its external segment equals the product of the other secant and its external segment.

533. If a tangent and a secant are drawn to a circle from an external point, the tangent is the mean proportional between the whole secant and its external segment.

534. The circumference of a circle equals the diameter times π.

535. The area of a circle equals π times the square of the radius.

536. In the same circle or in equal circles the length of an arc is to the circumference of the circle as the angular measure of the arc is to 360°.

537. In the same circle or in equal circles the area of a sector is to the area of the circle as the angular measure of the sector is to 360°.

Constructions

538. A given line segment or an angle can be bisected.

539. Through a given point one, and only one, perpendicular to a given line can be constructed.

540. Through a given external point one, and only one, parallel to a given line can be constructed.

541. Through a given point on or outside a given circle a tangent to the circle can be constructed.

542. A polygon can be transformed into a triangle (to have the same area).

Proportion

543. In any proportion the product of the means equals the product of the extremes.

544. If the product of one pair of numbers equals the product of another pair of numbers, either pair may be made the means and the other pair the extremes of a proportion.

545. A proportion may be transformed in any way that results in the same equality between the product of the means and the product of the extremes.

546. If four quantities are in proportion, they are in proportion by inversion or alternation.

547. In any proportion the sum of the first two terms is to the first (or the second) as the sum of the last two terms is to the third (or the fourth).

548. In a series of equal ratios the sum of any number of numerators is to the sum of their denominators as any numerator is to its denominator.

549. If the product of two variables is constant, one variable varies inversely as the other.

Outline for Short Course

Some teachers now devote less than a full semester to solid geometry and use the rest of the time on an integrated course involving some elements of analytical geometry. These teachers will find that DYNAMIC SOLID GEOMETRY can be readily adapted to their needs for two reasons: (1) The necessary propositions have been developed informally in a narrative style, making it possible for the student to learn the important facts without studying the proofs in detail. (2) There is an unusually large number of numerical exercises in concentrated units to form the backbone of this course.

In an abbreviated course the students are usually asked to prove formally a few representative theorems in the early part of the work so as to learn what a proof in solid geometry involves and to compare it with a proof in plane geometry. The outline below suggests the text material to emphasize. The theorems that are suitable for proof are starred (*). The number of numerical exercises available in connection with various units is also indicated.

CHAPTER 1 EMERGING FROM THE PLANE

		Time
Lesson 1, page 1	Plane Geometry §§ 1, 2	$\frac{1}{2}$–1 day
Lesson 2, page 5	Perspective §§ 3, 4	$\frac{1}{2}$–1 day
Lesson 3, page 10	Space Figures	0–1 day
Lesson 4, page 13	Properties of a Plane §§ 8, 10–13, 18	$\frac{1}{2}$–1 day
Lesson 5, page 17	Adapting Plane Geometry to Solid Figures §§ 20, 22, 23	0–1 day
Lesson 6, page 21	A Perpendicular to a Plane §§ 25*, 27, 30*; 4 numerical exercises	1–2 days
Lesson 7, page 29	Line Segments from a Point to a Plane §§ 38*, 43*, 46	$\frac{1}{2}$–1 day
Lesson 8, page 32	Two Perpendiculars to a Plane §§ 48*, 49*, 50*	1–2 days
Lesson 9, page 35	Summary § 52	1–2 days
	Total	5–12 days

CHAPTER 2 ASSEMBLING PLANES

Lesson 1, page 42	Parallel Planes §§ 54*, 57, 58, 61*	1–2 days
Lesson 2, page 45	A Line Parallel to a Plane §§ 64*, 66	$\frac{1}{2}$–1 day
Lesson 3, page 47	Parallel Lines §§ 67*, 68*, 69	$\frac{1}{2}$–1 day
Lesson 4, page 50	Perpendicular Planes §§ 82*, 85	1–2 days
Lesson 5, page 54	Projections §§ 88*, 91, 94, 96	1–2 days
Lesson 6, page 61	Figures Made by Three or More Planes §§ 99, 100; 6 numerical exercises	1–2 days
Lesson 7, page 65	Polyhedrons §§ 102, 103, 107	1–2 days
Lesson 8, page 75	Summary § 113	1–2 days
	Total	7–14 days

Chapter 3 TRANSLATING A LINE

Lesson 1, page 82	Formation of Prisms and Cylinders	½–1 day
Lesson 2, page 86	Properties of the Prism and the Cylinder	½–1 day
Lesson 3, page 91	Areas §§ 154, 165; 28 numerical exercises	2–3 days
Lesson 4, page 96	Volumes §§ 172, 176, 178, 184; 47 numerical exercises	2–4 days
Lesson 5, page 105	Summary § 185; 27 numerical exercises	2–4 days
	Total	7–13 days

Chapter 4 ROTATING A LINE

Lesson 1, page 111	Formation of Pyramids and Cones	1–1 day
Lesson 2, page 114	Properties of the Pyramid and the Cone § 216; 25 numerical exercises	1–2 days
Lesson 3, page 121	Areas §§ 218, 227; 30 numerical exercises	2–4 days
Lesson 4, page 126	Volumes §§ 232, 236, 239; 34 numerical exercises	2–4 days
Lesson 5, page 136	Summary § 243; 47 numerical exercises	2–4 days
	Total	8–15 days

Chapter 5 ROTATING A SEMICIRCLE

Lesson 1, page 144	The Sphere and Its Circles; 8 numerical exercises	½–1 day
Lesson 2, page 149	Inscribed and Circumscribed Spheres; 6 numerical exercises	½–1 day
Lesson 3, page 155	Area of a Sphere §§ 273, 281; 43 numerical exercises	2–4 days
Lesson 4, page 162	Volume of a Sphere §§ 290, 291; 67 numerical exercises	2–4 days
Lesson 5, page 172	Figures on a Sphere §§ 302, 304, 313, 321; 23 numerical exercises	0–2 days
Lesson 6, page 180	Polar Triangles § 327; 9 numerical exercises	0–1 day
Lesson 7, page 186	Congruent Spherical Triangles	0–1 day
Lesson 8, page 192	Areas of Spherical Triangles §§ 348–355; 25 numerical exercises	2–4 days
Lesson 9, page 198	Summary § 356; 48 numerical exercises	2–4 days
	Total	9–22 days
	Grand total	36–76 days

TABLE OF SINES, COSINES, AND TANGENTS

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000	90°	1.0000	.0000	

SQUARE ROOTS

N	0	1	2	3	4	5	6	7	8	9
1.0	1.000	1.005	1.010	1.015	1.020	1.025	1.030	1.034	1.039	1.044
1.1	1.049	1.054	1.058	1.063	1.068	1.072	1.077	1.082	1.086	1.091
1.2	1.095	1.100	1.105	1.109	1.114	1.118	1.122	1.127	1.131	1.136
1.3	1.140	1.145	1.149	1.153	1.158	1.162	1.166	1.170	1.175	1.179
1.4	1.183	1.187	1.192	1.196	1.200	1.204	1.208	1.212	1.217	1.221
1.5	1.225	1.229	1.233	1.237	1.241	1.245	1.249	1.253	1.257	1.261
1.6	1.265	1.269	1.273	1.277	1.281	1.285	1.288	1.292	1.296	1.300
1.7	1.304	1.308	1.311	1.315	1.319	1.323	1.327	1.330	1.334	1.338
1.8	1.342	1.345	1.349	1.353	1.356	1.360	1.364	1.367	1.371	1.375
1.9	1.378	1.382	1.386	1.389	1.393	1.396	1.400	1.404	1.407	1.411
2.0	1.414	1.418	1.421	1.425	1.428	1.432	1.435	1.439	1.442	1.446
2.1	1.449	1.453	1.456	1.459	1.463	1.466	1.470	1.473	1.476	1.480
2.2	1.483	1.487	1.490	1.493	1.497	1.500	1.503	1.507	1.510	1.513
2.3	1.517	1.520	1.523	1.526	1.530	1.533	1.536	1.539	1.543	1.546
2.4	1.549	1.552	1.556	1.559	1.562	1.565	1.568	1.572	1.575	1.578
2.5	1.581	1.584	1.587	1.591	1.594	1.597	1.600	1.603	1.606	1.609
2.6	1.612	1.616	1.619	1.622	1.625	1.628	1.631	1.634	1.637	1.640
2.7	1.643	1.646	1.649	1.652	1.655	1.658	1.661	1.664	1.667	1.670
2.8	1.673	1.676	1.679	1.682	1.685	1.688	1.691	1.694	1.697	1.700
2.9	1.703	1.706	1.709	1.712	1.715	1.718	1.720	1.723	1.726	1.729
3.0	1.732	1.735	1.738	1.741	1.744	1.746	1.749	1.752	1.755	1.758
3.1	1.761	1.764	1.766	1.769	1.772	1.775	1.778	1.780	1.783	1.786
3.2	1.789	1.792	1.794	1.797	1.800	1.803	1.806	1.808	1.811	1.814
3.3	1.817	1.819	1.822	1.825	1.828	1.830	1.833	1.836	1.838	1.841
3.4	1.844	1.847	1.849	1.852	1.855	1.857	1.860	1.863	1.865	1.868
3.5	1.871	1.873	1.876	1.879	1.881	1.884	1.887	1.889	1.892	1.895
3.6	1.897	1.900	1.903	1.905	1.908	1.910	1.913	1.916	1.918	1.921
3.7	1.924	1.926	1.929	1.931	1.934	1.936	1.939	1.942	1.944	1.947
3.8	1.949	1.952	1.954	1.957	1.960	1.962	1.965	1.967	1.970	1.972
3.9	1.975	1.977	1.980	1.982	1.985	1.987	1.990	1.992	1.995	1.997
4.0	2.000	2.002	2.005	2.007	2.010	2.012	2.015	2.017	2.020	2.022
4.1	2.025	2.027	2.030	2.032	2.035	2.037	2.040	2.042	2.045	2.047
4.2	2.049	2.052	2.054	2.057	2.059	2.062	2.064	2.066	2.069	2.071
4.3	2.074	2.076	2.078	2.081	2.083	2.086	2.088	2.090	2.093	2.095
4.4	2.098	2.100	2.102	2.105	2.107	2.110	2.112	2.114	2.117	2.119
4.5	2.121	2.124	2.126	2.128	2.131	2.133	2.135	2.138	2.140	2.142
4.6	2.145	2.147	2.149	2.152	2.154	2.156	2.159	2.161	2.163	2.166
4.7	2.168	2.170	2.173	2.175	2.177	2.179	2.182	2.184	2.186	2.189
4.8	2.191	2.193	2.195	2.198	2.200	2.202	2.205	2.207	2.209	2.211
4.9	2.214	2.216	2.218	2.220	2.223	2.225	2.227	2.229	2.232	2.234
5.0	2.236	2.238	2.241	2.243	2.245	2.247	2.249	2.252	2.254	2.256
5.1	2.258	2.261	2.263	2.265	2.267	2.269	2.272	2.274	2.276	2.278
5.2	2.280	2.283	2.285	2.287	2.289	2.291	2.293	2.296	2.298	2.300
5.3	2.302	2.304	2.307	2.309	2.311	2.313	2.315	2.317	2.319	2.322
5.4	2.324	2.326	2.328	2.330	2.332	2.335	2.337	2.339	2.341	2.343

Note: To find the square root of a number not between 1 and 100, shift the decimal point. Thus, $\sqrt{175} = \sqrt{1.75} \times 10$; $\sqrt{.175} = \sqrt{17.5} \div 10$.

SQUARE ROOTS

N	0	1	2	3	4	5	6	7	8	9
5.5	2.345	2.347	2.349	2.352	2.354	2.356	2.358	2.360	2.362	2.364
5.6	2.366	2.369	2.371	2.373	2.375	2.377	2.379	2.381	2.383	2.385
5.7	2.387	2.390	2.392	2.394	2.396	2.398	2.400	2.402	2.404	2.406
5.8	2.408	2.410	2.412	2.415	2.417	2.419	2.421	2.423	2.425	2.427
5.9	2.429	2.431	2.433	2.435	2.437	2.439	2.441	2.443	2.445	2.447
6.0	2.449	2.452	2.454	2.456	2.458	2.460	2.462	2.464	2.466	2.468
6.1	2.470	2.472	2.474	2.476	2.478	2.480	2.482	2.484	2.486	2.488
6.2	2.490	2.492	2.494	2.496	2.498	2.500	2.502	2.504	2.506	2.508
6.3	2.510	2.512	2.514	2.516	2.518	2.520	2.522	2.524	2.526	2.528
6.4	2.530	2.532	2.534	2.536	2.538	2.540	2.542	2.544	2.546	2.548
6.5	2.550	2.551	2.553	2.555	2.557	2.559	2.561	2.563	2.565	2.567
6.6	2.569	2.571	2.573	2.575	2.577	2.579	2.581	2.583	2.585	2.587
6.7	2.588	2.590	2.592	2.594	2.596	2.598	2.600	2.602	2.604	2.606
6.8	2.608	2.610	2.612	2.613	2.615	2.617	2.619	2.621	2.623	2.625
6.9	2.627	2.629	2.631	2.632	2.634	2.636	2.638	2.640	2.642	2.644
7.0	2.646	2.648	2.650	2.651	2.653	2.655	2.657	2.659	2.661	2.663
7.1	2.665	2.666	2.668	2.670	2.672	2.674	2.676	2.678	2.680	2.681
7.2	2.683	2.685	2.687	2.689	2.691	2.693	2.694	2.696	2.698	2.700
7.3	2.702	2.704	2.706	2.707	2.709	2.711	2.713	2.715	2.717	2.718
7.4	2.720	2.722	2.724	2.726	2.728	2.729	2.731	2.733	2.735	2.737
7.5	2.739	2.740	2.742	2.744	2.746	2.748	2.750	2.751	2.753	2.755
7.6	2.757	2.759	2.760	2.762	2.764	2.766	2.768	2.769	2.771	2.773
7.7	2.775	2.777	2.778	2.780	2.782	2.784	2.786	2.787	2.789	2.791
7.8	2.793	2.795	2.796	2.798	2.800	2.802	2.804	2.805	2.807	2.809
7.9	2.811	2.812	2.814	2.816	2.818	2.820	2.821	2.823	2.825	2.827
8.0	2.828	2.830	2.832	2.834	2.835	2.837	2.839	2.841	2.843	2.844
8.1	2.846	2.848	2.850	2.851	2.853	2.855	2.857	2.858	2.860	2.862
8.2	2.864	2.865	2.867	2.869	2.871	2.872	2.874	2.876	2.877	2.879
8.3	2.881	2.883	2.884	2.886	2.888	2.890	2.891	2.893	2.895	2.897
8.4	2.898	2.900	2.902	2.903	2.905	2.907	2.909	2.910	2.912	2.914
8.5	2.915	2.917	2.919	2.921	2.922	2.924	2.926	2.927	2.929	2.931
8.6	2.933	2.934	2.936	2.938	2.939	2.941	2.943	2.944	2.946	2.948
8.7	2.950	2.951	2.953	2.955	2.956	2.958	2.960	2.961	2.963	2.965
8.8	2.966	2.968	2.970	2.972	2.973	2.975	2.977	2.978	2.980	2.982
8.9	2.983	2.985	2.987	2.988	2.990	2.992	2.993	2.995	2.997	2.998
9.0	3.000	3.002	3.003	3.005	3.007	3.008	3.010	3.012	3.013	3.015
9.1	3.017	3.018	3.020	3.022	3.023	3.025	3.027	3.028	3.030	3.032
9.2	3.033	3.035	3.036	3.038	3.040	3.041	3.043	3.045	3.046	3.048
9.3	3.050	3.051	3.053	3.055	3.056	3.058	3.059	3.061	3.063	3.064
9.4	3.066	3.068	3.069	3.071	3.072	3.074	3.076	3.077	3.079	3.081
9.5	3.082	3.084	3.085	3.087	3.089	3.090	3.092	3.094	3.095	3.097
9.6	3.098	3.100	3.102	3.103	3.105	3.106	3.108	3.110	3.111	3.113
9.7	3.114	3.116	3.118	3.119	3.121	3.122	3.124	3.126	3.127	3.129
9.8	3.130	3.132	3.134	3.135	3.137	3.138	3.140	3.142	3.143	3.145
9.9	3.146	3.148	3.150	3.151	3.153	3.154	3.156	3.158	3.159	3.161

SQUARE ROOTS

N	0	1	2	3	4	5	6	7	8	9
10.	3.162	3.178	3.194	3.209	3.225	3.240	3.256	3.271	3.286	3.302
11.	3.317	3.332	3.347	3.362	3.376	3.391	3.406	3.421	3.435	3.450
12.	3.464	3.479	3.493	3.507	3.521	3.536	3.550	3.564	3.578	3.592
13.	3.606	3.619	3.633	3.647	3.661	3.674	3.688	3.701	3.715	3.728
14.	3.742	3.755	3.768	3.782	3.795	3.808	3.821	3.834	3.847	3.860
15.	3.873	3.886	3.899	3.912	3.924	3.937	3.950	3.962	3.975	3.987
16.	4.000	4.012	4.025	4.037	4.050	4.062	4.074	4.087	4.099	4.111
17.	4.123	4.135	4.147	4.159	4.171	4.183	4.195	4.207	4.219	4.231
18.	4.243	4.254	4.266	4.278	4.290	4.301	4.313	4.324	4.336	4.347
19.	4.359	4.370	4.382	4.393	4.405	4.416	4.427	4.438	4.450	4.461
20.	4.472	4.483	4.494	4.506	4.517	4.528	4.539	4.550	4.561	4.572
21.	4.583	4.593	4.604	4.615	4.626	4.637	4.648	4.658	4.669	4.680
22.	4.690	4.701	4.712	4.722	4.733	4.743	4.754	4.764	4.775	4.785
23.	4.796	4.806	4.817	4.827	4.837	4.848	4.858	4.868	4.879	4.889
24.	4.899	4.909	4.919	4.930	4.940	4.950	4.960	4.970	4.980	4.990
25.	5.000	5.010	5.020	5.030	5.040	5.050	5.060	5.070	5.079	5.089
26.	5.099	5.109	5.119	5.128	5.138	5.148	5.158	5.167	5.177	5.187
27.	5.196	5.206	5.215	5.225	5.235	5.244	5.254	5.263	5.273	5.282
28.	5.292	5.301	5.310	5.320	5.329	5.339	5.348	5.357	5.367	5.376
29.	5.385	5.394	5.404	5.413	5.422	5.431	5.441	5.450	5.459	5.468
30.	5.477	5.486	5.495	5.505	5.514	5.523	5.532	5.541	5.550	5.559
31.	5.568	5.577	5.586	5.595	5.604	5.612	5.621	5.630	5.639	5.648
32.	5.657	5.666	5.675	5.683	5.692	5.701	5.710	5.718	5.727	5.736
33.	5.745	5.753	5.762	5.771	5.779	5.788	5.797	5.805	5.814	5.822
34.	5.831	5.840	5.848	5.857	5.865	5.874	5.882	5.891	5.899	5.908
35.	5.916	5.925	5.933	5.941	5.950	5.958	5.967	5.975	5.983	5.992
36.	6.000	6.008	6.017	6.025	6.033	6.042	6.050	6.058	6.066	6.075
37.	6.083	6.091	6.099	6.107	6.116	6.124	6.132	6.140	6.148	6.156
38.	6.164	6.173	6.181	6.189	6.197	6.205	6.213	6.221	6.229	6.237
39.	6.245	6.253	6.261	6.269	6.277	6.285	6.293	6.301	6.309	6.317
40.	6.325	6.332	6.340	6.348	6.356	6.364	6.372	6.380	6.387	6.395
41.	6.403	6.411	6.419	6.427	6.434	6.442	6.450	6.458	6.465	6.473
42.	6.481	6.488	6.496	6.504	6.512	6.519	6.527	6.535	6.542	6.550
43.	6.557	6.565	6.573	6.580	6.588	6.595	6.603	6.611	6.618	6.626
44.	6.633	6.641	6.648	6.656	6.663	6.671	6.678	6.686	6.693	6.701
45.	6.708	6.716	6.723	6.731	6.738	6.745	6.753	6.760	6.768	6.775
46.	6.782	6.790	6.797	6.804	6.812	6.819	6.826	6.834	6.841	6.848
47.	6.856	6.863	6.870	6.877	6.885	6.892	6.899	6.907	6.914	6.921
48.	6.928	6.935	6.943	6.950	6.957	6.964	6.971	6.979	6.986	6.993
49.	7.000	7.007	7.014	7.021	7.029	7.036	7.043	7.050	7.057	7.064
50.	7.071	7.078	7.085	7.092	7.099	7.106	7.113	7.120	7.127	7.134
51.	7.141	7.148	7.155	7.162	7.169	7.176	7.183	7.190	7.197	7.204
52.	7.211	7.218	7.225	7.232	7.239	7.246	7.253	7.259	7.266	7.273
53.	7.280	7.287	7.294	7.301	7.308	7.314	7.321	7.328	7.335	7.342
54.	7.348	7.355	7.362	7.369	7.376	7.382	7.389	7.396	7.403	7.409

SQUARE ROOTS

N	0	1	2	3	4	5	6	7	8	9
55.	7.416	7.423	7.430	7.436	7.443	7.450	7.457	7.463	7.470	7.477
56.	7.483	7.490	7.497	7.503	7.510	7.517	7.523	7.530	7.537	7.543
57.	7.550	7.556	7.563	7.570	7.576	7.583	7.589	7.596	7.603	7.609
58.	7.616	7.622	7.629	7.635	7.642	7.649	7.655	7.662	7.668	7.675
59.	7.681	7.688	7.694	7.701	7.707	7.714	7.720	7.727	7.733	7.740
60.	7.746	7.752	7.759	7.765	7.772	7.778	7.785	7.791	7.797	7.804
61.	7.810	7.817	7.823	7.829	7.836	7.842	7.849	7.855	7.861	7.868
62.	7.874	7.880	7.887	7.893	7.899	7.906	7.912	7.918	7.925	7.931
63.	7.937	7.944	7.950	7.956	7.962	7.969	7.975	7.981	7.987	7.994
64.	8.000	8.006	8.012	8.019	8.025	8.031	8.037	8.044	8.050	8.056
65.	8.062	8.068	8.075	8.081	8.087	8.093	8.099	8.106	8.112	8.118
66.	8.124	8.130	8.136	8.142	8.149	8.155	8.161	8.167	8.173	8.179
67.	8.185	8.191	8.198	8.204	8.210	8.216	8.222	8.228	8.234	8.240
68.	8.246	8.252	8.258	8.264	8.270	8.276	8.283	8.289	8.295	8.301
69.	8.307	8.313	8.319	8.325	8.331	8.337	8.343	8.349	8.355	8.361
70.	8.367	8.373	8.379	8.385	8.390	8.396	8.402	8.408	8.414	8.420
71.	8.426	8.432	8.438	8.444	8.450	8.456	8.462	8.468	8.473	8.479
72.	8.485	8.491	8.497	8.503	8.509	8.515	8.521	8.526	8.532	8.538
73.	8.544	8.550	8.556	8.562	8.567	8.573	8.579	8.585	8.591	8.597
74.	8.602	8.608	8.614	8.620	8.626	8.631	8.637	8.643	8.649	8.654
75.	8.660	8.666	8.672	8.678	8.683	8.689	8.695	8.701	8.706	8.712
76.	8.718	8.724	8.729	8.735	8.741	8.746	8.752	8.758	8.764	8.769
77.	8.775	8.781	8.786	8.792	8.798	8.803	8.809	8.815	8.820	8.826
78.	8.832	8.837	8.843	8.849	8.854	8.860	8.866	8.871	8.877	8.883
79.	8.888	8.894	8.899	8.905	8.911	8.916	8.922	8.927	8.933	8.939
80.	8.944	8.950	8.955	8.961	8.967	8.972	8.978	8.983	8.989	8.994
81.	9.000	9.006	9.011	9.017	9.022	9.028	9.033	9.039	9.044	9.050
82.	9.055	9.061	9.066	9.072	9.077	9.083	9.088	9.094	9.099	9.105
83.	9.110	9.116	9.121	9.127	9.132	9.138	9.143	9.149	9.154	9.160
84.	9.165	9.171	9.176	9.182	9.187	9.192	9.198	9.203	9.209	9.214
85.	9.220	9.225	9.230	9.236	9.241	9.247	9.252	9.257	9.263	9.268
86.	9.274	9.279	9.284	9.290	9.295	9.301	9.306	9.311	9.317	9.322
87.	9.327	9.333	9.338	9.343	9.349	9.354	9.359	9.365	9.370	9.375
88.	9.381	9.386	9.391	9.397	9.402	9.407	9.413	9.418	9.423	9.429
89.	9.434	9.439	9.445	9.450	9.455	9.460	9.466	9.471	9.476	9.482
90.	9.487	9.492	9.497	9.503	9.508	9.513	9.518	9.524	9.529	9.534
91.	9.539	9.545	9.550	9.555	9.560	9.566	9.571	9.576	9.581	9.586
92.	9.592	9.597	9.602	9.607	9.612	9.618	9.623	9.628	9.633	9.638
93.	9.644	9.649	9.654	9.659	9.664	9.670	9.675	9.680	9.685	9.690
94.	9.695	9.701	9.706	9.711	9.716	9.721	9.726	9.731	9.737	9.742
95.	9.747	9.752	9.757	9.762	9.767	9.772	9.778	9.783	9.788	9.793
96.	9.798	9.803	9.808	9.813	9.818	9.823	9.829	9.834	9.839	9.844
97.	9.849	9.854	9.859	9.864	9.869	9.874	9.879	9.884	9.889	9.894
98.	9.899	9.905	9.910	9.915	9.920	9.925	9.930	9.935	9.940	9.945
99.	9.950	9.955	9.960	9.965	9.970	9.975	9.980	9.985	9.990	9.995

N	0	1	2	3	4	5	6	7	8	9
1.0	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374
1.1	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755
1.2	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106
1.3	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430
1.4	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732
1.5	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014
1.6	.2041	.2068	.2095	.2122	.2148	.2175	.2201	.2227	.2253	.2279
1.7	.2304	.2330	.2355	.2380	.2405	.2430	.2455	.2480	.2504	.2529
1.8	.2553	.2577	.2601	.2625	.2648	.2672	.2695	.2718	.2742	.2765
1.9	.2788	.2810	.2833	.2856	.2878	.2900	.2923	.2945	.2967	.2989
2.0	.3010	.3032	.3054	.3075	.3096	.3118	.3139	.3160	.3181	.3201
2.1	.3222	.3243	.3263	.3284	.3304	.3324	.3345	.3365	.3385	.3404
2.2	.3424	.3444	.3464	.3483	.3502	.3522	.3541	.3560	.3579	.3598
2.3	.3617	.3636	.3655	.3674	.3692	.3711	.3729	.3747	.3766	.3784
2.4	.3802	.3820	.3838	.3856	.3874	.3892	.3909	.3927	.3945	.3962
2.5	.3979	.3997	.4014	.4031	.4048	.4065	.4082	.4099	.4116	.4133
2.6	.4150	.4166	.4183	.4200	.4216	.4232	.4249	.4265	.4281	.4298
2.7	.4314	.4330	.4346	.4362	.4378	.4393	.4409	.4425	.4440	.4456
2.8	.4472	.4487	.4502	.4518	.4533	.4548	.4564	.4579	.4594	.4609
2.9	.4624	.4639	.4654	.4669	.4683	.4698	.4713	.4728	.4742	.4757
3.0	.4771	.4786	.4800	.4814	.4829	.4843	.4857	.4871	.4886	.4900
3.1	.4914	.4928	.4942	.4955	.4969	.4983	.4997	.5011	.5024	.5038
3.2	.5051	.5065	.5079	.5092	.5105	.5119	.5132	.5145	.5159	.5172
3.3	.5185	.5198	.5211	.5224	.5237	.5250	.5263	.5276	.5289	.5302
3.4	.5315	.5328	.5340	.5353	.5366	.5378	.5391	.5403	.5416	.5428
3.5	.5441	.5453	.5465	.5478	.5490	.5502	.5514	.5527	.5539	.5551
3.6	.5563	.5575	.5587	.5599	.5611	.5623	.5635	.5647	.5658	.5670
3.7	.5682	.5694	.5705	.5717	.5729	.5740	.5752	.5763	.5775	.5786
3.8	.5798	.5809	.5821	.5832	.5843	.5855	.5866	.5877	.5888	.5899
3.9	.5911	.5922	.5933	.5944	.5955	.5966	.5977	.5988	.5999	.6010
4.0	.6021	.6031	.6042	.6053	.6064	.6075	.6085	.6096	.6107	.6117
4.1	.6128	.6138	.6149	.6160	.6170	.6180	.6191	.6201	.6212	.6222
4.2	.6232	.6243	.6253	.6263	.6274	.6284	.6294	.6304	.6314	.6325
4.3	.6335	.6345	.6355	.6365	.6375	.6385	.6395	.6405	.6415	.6425
4.4	.6435	.6444	.6454	.6464	.6474	.6484	.6493	.6503	.6513	.6522
4.5	.6532	.6542	.6551	.6561	.6571	.6580	.6590	.6599	.6609	.6618
4.6	.6628	.6637	.6646	.6656	.6665	.6675	.6684	.6693	.6702	.6712
4.7	.6721	.6730	.6739	.6749	.6758	.6767	.6776	.6785	.6794	.6803
4.8	.6812	.6821	.6830	.6839	.6848	.6857	.6866	.6875	.6884	.6893
4.9	.6902	.6911	.6920	.6928	.6937	.6946	.6955	.6964	.6972	.6981
5.0	.6990	.6998	.7007	.7016	.7024	.7033	.7042	.7050	.7059	.7067
5.1	.7076	.7084	.7093	.7101	.7110	.7118	.7126	.7135	.7143	.7152
5.2	.7160	.7168	.7177	.7185	.7193	.7202	.7210	.7218	.7226	.7235
5.3	.7243	.7251	.7259	.7267	.7275	.7284	.7292	.7300	.7308	.7316
5.4	.7324	.7332	.7340	.7348	.7356	.7364	.7372	.7380	.7388	.7396
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
5.5	.7404	.7412	.7419	.7427	.7435	.7443	.7451	.7459	.7466	.7474
5.6	.7482	.7490	.7497	.7505	.7513	.7520	.7528	.7536	.7543	.7551
5.7	.7559	.7566	.7574	.7582	.7589	.7597	.7604	.7612	.7619	.7627
5.8	.7634	.7642	.7649	.7657	.7664	.7672	.7679	.7686	.7694	.7701
5.9	.7709	.7716	.7723	.7731	.7738	.7745	.7752	.7760	.7767	.7774
6.0	.7782	.7789	.7796	.7803	.7810	.7818	.7825	.7832	.7839	.7846
6.1	.7853	.7860	.7868	.7875	.7882	.7889	.7896	.7903	.7910	.7917
6.2	.7924	.7931	.7938	.7945	.7952	.7959	.7966	.7973	.7980	.7987
6.3	.7993	.8000	.8007	.8014	.8021	.8028	.8035	.8041	.8048	.8055
6.4	.8062	.8069	.8075	.8082	.8089	.8096	.8102	.8109	.8116	.8122
6.5	.8129	.8136	.8142	.8149	.8156	.8162	.8169	.8176	.8182	.8189
6.6	.8195	.8202	.8209	.8215	.8222	.8228	.8235	.8241	.8248	.8254
6.7	.8261	.8267	.8274	.8280	.8287	.8293	.8299	.8306	.8312	.8319
6.8	.8325	.8331	.8338	.8344	.8351	.8357	.8363	.8370	.8376	.8382
6.9	.8388	.8395	.8401	.8407	.8414	.8420	.8426	.8432	.8439	.8445
7.0	.8451	.8457	.8463	.8470	.8476	.8482	.8488	.8494	.8500	.8506
7.1	.8513	.8519	.8525	.8531	.8537	.8543	.8549	.8555	.8561	.8567
7.2	.8573	.8579	.8585	.8591	.8597	.8603	.8609	.8615	.8621	.8627
7.3	.8633	.8639	.8645	.8651	.8657	.8663	.8669	.8675	.8681	.8686
7.4	.8692	.8698	.8704	.8710	.8716	.8722	.8727	.8733	.8739	.8745
7.5	.8751	.8756	.8762	.8768	.8774	.8779	.8785	.8791	.8797	.8802
7.6	.8808	.8814	.8820	.8825	.8831	.8837	.8842	.8848	.8854	.8859
7.7	.8865	.8871	.8876	.8882	.8887	.8893	.8899	.8904	.8910	.8915
7.8	.8921	.8927	.8932	.8938	.8943	.8949	.8954	.8960	.8965	.8971
7.9	.8976	.8982	.8987	.8993	.8998	.9004	.9009	.9015	.9020	.9025
8.0	.9031	.9036	.9042	.9047	.9053	.9058	.9063	.9069	.9074	.9079
8.1	.9085	.9090	.9096	.9101	.9106	.9112	.9117	.9122	.9128	.9133
8.2	.9138	.9143	.9149	.9154	.9159	.9165	.9170	.9175	.9180	.9186
8.3	.9191	.9196	.9201	.9206	.9212	.9217	.9222	.9227	.9232	.9238
8.4	.9243	.9248	.9253	.9258	.9263	.9269	.9274	.9279	.9284	.9289
8.5	.9294	.9299	.9304	.9309	.9315	.9320	.9325	.9330	.9335	.9340
8.6	.9345	.9350	.9355	.9360	.9365	.9370	.9375	.9380	.9385	.9390
8.7	.9395	.9400	.9405	.9410	.9415	.9420	.9425	.9430	.9435	.9440
8.8	.9445	.9450	.9455	.9460	.9465	.9469	.9474	.9479	.9484	.9489
8.9	.9494	.9499	.9504	.9509	.9513	.9518	.9523	.9528	.9533	.9538
9.0	.9542	.9547	.9552	.9557	.9562	.9566	.9571	.9576	.9581	.9586
9.1	.9590	.9595	.9600	.9605	.9609	.9614	.9619	.9624	.9628	.9633
9.2	.9638	.9643	.9647	.9652	.9657	.9661	.9666	.9671	.9675	.9680
9.3	.9685	.9689	.9694	.9699	.9703	.9708	.9713	.9717	.9722	.9727
9.4	.9731	.9736	.9741	.9745	.9750	.9754	.9759	.9763	.9768	.9773
9.5	.9777	.9782	.9786	.9791	.9795	.9800	.9805	.9809	.9814	.9818
9.6	.9823	.9827	.9832	.9836	.9841	.9845	.9850	.9854	.9859	.9863
9.7	.9868	.9872	.9877	.9881	.9886	.9890	.9894	.9899	.9903	.9908
9.8	.9912	.9917	.9921	.9926	.9930	.9934	.9939	.9943	.9948	.9952
9.9	.9956	.9961	.9965	.9969	.9974	.9978	.9983	.9987	.9991	.9996
N	0	1	2	3	4	5	6	7	8	9

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